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Stability, bifurcation and post-critical behavior of a homogeneously deformed incompressible isotropic elastic parallelepiped subject to dead-load surface tractions

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ABSTRACT

We study the equilibrium homogeneous deformations of a homogeneous parallelepiped made of an arbitrary incompressible, isotropic elastic material and subject to a distribution of dead-load surface tractions corresponding to an equibiaxial tensile stress state accompanied by an orthogonal uniaxial compression of the same amount. We show that only two classes of homogeneous equilibrium solutions are possible, namely *symmetric* deformations, characterized by two equal principal stretches, and *asymmetric* deformations, with all different principal stretches. Following the classical energy-stability criterion, we then find necessary and sufficient conditions for both symmetric and asymmetric equilibrium deformations to be weak relative minimizers of the total potential energy. Finally, we analyze the mechanical response of a parallelepiped made of an incompressible Mooney–Rivlin material in a monotonic dead loading process starting from the unloaded state. As a major result, we model the actual occurrence of a bifurcation from a primary branch of locally stable symmetric deformations to a secondary, post-critical branch of locally stable asymmetric solutions.

1. Introduction

In the present paper we study the equilibrium deformations of an incompressible homogeneous, isotropic elastic parallelepiped subject in its reference configuration to a uniform distribution of dead-load surface tractions orthogonal to its faces, with magnitude $s>0$ on two pairs of faces and magnitude $-s$ on the remaining two faces. We neglect body forces. This kind of traction boundary conditions should be useful in describing meaningful physical situations, such as the multiaxial loading experimental response of specimens subject to a equibiaxial tension accompanied by a uniaxial compressive force of the same amount.

We focus on homogeneous equilibrium deformations, that is those for which the deformation gradient is constant, and we study their local stability, the possibility of bifurcations and the post-critical response of the body as the load parameter s varies. In particular, the equilibrium and the local stability analyses of homogeneous deformations are performed for an arbitrary incompressible isotropic elastic solid, whereas the study on bifurcations and post-critical behavior is performed for the particular case of the classical Mooney–Rivlin constitutive model.

Stability and bifurcation analyses for elastic bodies held in equilibrium by uniform dead-loads have a long history in non-linear elasticity and they have been employed to explore the features of the equilibrium solutions ever since Rivlin [1,2] studied homogeneous deformations of a cube of neo-Hookean material. Subsequently, other fundamental investigations on these topics have been performed, for instance, by Bromberg [3], by Sawyers [4], by Ball and Schaeffer [5] for an arbitrary isotropic incompressible material and by MacSithigh and Chen [6].

Recently, new interesting studies [7] dealing with the non-linear stability analysis of inhomogeneous deformations for three-dimensional incompressible hyperelastic bodies and its connection to bifurcation issues suggested the importance of combined stability and bifurcation analyses, in order to provide a setting both for better understanding the behavior of complex systems for engineering applications. What one usually wishes to model in such situations may be referred, for example, to the paradigmatic case of the Euler beam, where the primary stable mode becomes unstable in correspondence of a bifurcation point, and the new bifurcating mode is characterized by a stable post-critical response. It is worth to note that for one-dimensional problems heuristic approaches based on formal perturbative techniques have been successfully adopted both in static and dynamic

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bifurcation analyses (see, e.g., [8]) and, in some cases, they may give suggestions for the formulation of rigorous treatments.

Although in a number of studies the Euler beam paradigm is reproduced (see, e.g., [9] and [10]), in the framework of three-dimensional non-linear elasticity one may find few cases wherein the loss of stability of a fundamental mode is accompanied by the occurrence of a bifurcating mode characterized by a stable post-critical response. Indeed, one may find analyses dealing with non-uniqueness of solutions without a loss of stability, like in phenomena involving stable coexistent deformations in stress-induced phase transformations (see, e.g., [11]), or with situations characterized by a loss of stability due to softening without the occurrence of a new bifurcating mode. Nevertheless, there are examples which suggest a close relationship between bifurcation and stability issues, at least restricted to certain particular problems. For instance, a close connection in non-linear elasticity may be encountered when studying "small on large" problems through the classical method of adjacent equilibria, i.e., when one checks if there exist other equilibrium solutions close to a given primary equilibrium state through the analysis of the equilibrium equations linearized around the deformed configuration.

In this vein, the Hadamard energetic criterion of infinitesimal stability (cf. [12], §68 bis) is a classical and widely used tool for testing the stability of an equilibrium state, since the violation of the so-called Hadamard stability condition plays the role of an indicator of *possible* bifurcations from the primary equilibrium configuration. Indeed, in a monotonic loading process governed by a loading parameter it is usually assumed that the body remains in the fundamental equilibrium state (no bifurcation allowed) until the Hadamard stability inequality is strictly satisfied. Then, possible bifurcation modes may arise at the critical load, which in this context is defined as the value of the load parameter which first renders the Hadamard functional zero.

These considerations show that stability in the Hadamard sense requires the determination of the sign of the Hadamard functional, which usually is not a simple matter. In certain cases it may be very helpful developing a lower bound estimate for the Hadamard functional, with the major goal of seeking a lower bound estimate for the critical load, below which the Hadamard stability condition is definitely satisfied. This strategy may be useful in bifurcation problems: one may evaluate the critical load corresponding to a special bifurcation mode by simply studying the sign of the Hadamard functional on the corresponding class of incremental displacements. Since this analysis is performed within a restricted class of possible adjacent displacements, the determined critical load may be seen as an upper bound estimate of the actual critical load. Then, one may wonder if such a special superposed solution is actually the first bifurcation mode (among other bifurcation modes belonging to the whole class of kinematically admissible incremental displacements) which occurs during the considered loading process. If, as usual, it is prohibitive to check the sign of the Hadamard functional on the whole class of incremental displacements, a possible answer may be obtained by checking whether the gap between the critical load evaluated for the special bifurcation mode (upper bound) and the lower bound estimate for the actual critical load is sufficiently small or not.

In the recent literature new lower bound estimates for the critical load both for compressible or incompressible elastic bodies have been proposed (see, e.g., [13,14,15]). These studies are motivated by the need of seeking "better quality" estimates which may narrow the "distance" between the load below which bifurcations are not possible (lower bound estimate) and the value of the actual critical load (or the value of an upper bound estimate of the critical load). In our previous papers [13,14] we have determined lower bound estimates of the Hadamard functional based on Korn's inequality either for compressible or incompressible hyperelastic solids. In particular, the effectiveness of the estimates proposed in [13] has been highlighted with reference to the special bifurcation problem from a homogeneously deformed state developed in [16]. We believe that the procedure in [13] may be useful for further developing the analysis of possible bifurcations from inhomogeneous deformations developed in [17].

The above considerations concern examples of strategies for checking if a primary stable deformation becomes unstable and if a bifurcation is possible, but such methods do not help to assert if a bifurcation mode actually occurs or, in other words, if actually there are local branches of post-bifurcating solutions. One may refer to [18] in order to find conditions for the existence of local branches of bifurcating solutions within the elliptic range, namely the classical strong ellipticity condition and the boundary complementing conditions for the fourth-order incremental elasticity tensor field which rules the equilibrium problem linearized around the bifurcation point.

We point out that the analysis in the present paper yields a (not common) explicit example within three-dimensional non-linear elasticity concerning all the topics above mentioned, specifically the bifurcation from a locally stable primary deformation to a secondary bifurcation characterized by a locally stable post-critical branch.

The paper is organized as follows.

In Section 2, we specify the particular form of the homogeneous dead-load tractions on the boundary and then we study the equilibrium of *homogeneous* deformations for a parallelepiped made of an arbitrary incompressible, isotropic elastic material. We determine the representation forms of the two possible classes of homogeneous solutions and, by applying a result in [19], we show that only one of these two classes can be considered as a possible local minimizer of the total potential energy. In particular, for such class of possible equilibrium solutions we find that two subclasses are allowed, respectively named *symmetric* solutions, having two equal principal stretches, and *asymmetric* solutions, with all different principal stretches.

In Section 3, we study the local stability of the homogeneous equilibrium deformations determined in Section 2 for an arbitrary incompressible isotropic elastic material. According to the energy stability criterion, a local stable deformation is identified as a weak relative minimizer of the total potential energy functional among all the admissible isochoric deformations. Our analysis, which is facilitated both by the special form of the boundary loading and by the assumption of focusing on homogeneous deformations, allows us to replace an integral inequality by an algebraic inequality which leads to a set of *four* necessary and sufficient conditions for the stability of the symmetric equilibrium deformations and to a set of *two* necessary and sufficient conditions for the stability of the asymmetric equilibrium solutions. More specifically, such conditions involve the principal Cauchy stresses and the principal stretches, which are determined from the equilibrium solutions and from the boundary loading.

Finally, in Section 4 we study the mechanical response of a parallelepiped made of an incompressible Mooney–Rivlin material and subject to a monotonic dead loading process starting from the unloaded state, with boundary loading as described in Section 2. We first observe that for small values of the loading parameter only symmetric solutions are allowed, whereas asymmetric solutions may occur only sufficiently far from the reference configuration. In particular, we show that symmetric solutions are the unique locally stable homogenous deformations until the load reaches a critical value, in correspondence of which the symmetric solutions lose their uniqueness, and a bifurcation into asymmetric solutions may occur. Then, by further increasing the load beyond its critical value, the symmetric solutions are no more locally