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Magnetohydrodynamic thermal instability in a conducting fluid layer with throughflow

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ABSTRACT

An analysis is made of the effect of vertical throughflow on the onset of thermal convection in a horizontal layer of an electrically conducting fluid contained between two rigid permeable plates and heated from below in the presence of a uniform vertical magnetic field. The constant throughflow is measured by Peclet number Q and at both boundaries heat flux is held constant. It is found that when both boundaries are perfectly electrically conducting, the critical value of Rayleigh number R_t^c at the onset of steady convection increases with increase in Q for given values of the magnetic parameter R_h, the Prandtl number p_1 and the magnetic Prandtl number p_2 with $p_1 > p_2$. It is observed that the magnetic field inhibits the onset of steady convection. The analysis further reveals that R_t^c is independent of the sign of Q. When the lower plate is electrically non-conducting and the upper plate is perfectly electrically conducting, R_t^c is greater than the corresponding value of R_t^c for perfectly conducting plates for given values of Q, R_h, p_1 and p_2. It is also found that the positive throughflow ($Q > 0$) is more stabilizing than the negative one ($Q < 0$). The results are exemplified by considering some realistic cases e.g., liquid sodium and gallium.

1. Introduction

It is well known that when a horizontal layer of viscous fluid is heated uniformly from below in a gravitational field, the basic state becomes unstable and steady convection sets in when the Rayleigh number R exceeds a certain critical value R_c. Determination of this criterion for the onset of convection is a classical problem associated with Rayleigh and Bénard. A comprehensive account of thermal instability is given in Chandrasekhar's monograph [1]. In this problem there is no flow of fluid across the horizontal boundaries. Shvartsblat [2,3] investigated a modified problem, where the boundaries are permeable, and there is a vertical constant throughflow caused by injection at one boundary and sucking out fluid at the other. Additional references on the works of Shvartsblat may be found in the book by Gershuni and Zhukhovitsky [4]. The problem is of interest because it gives rise to the possibility of controlling the thermal instability by adjustment of vertical throughflow as well as for its relevance to meteorology. In an effort to explore the relationship of cellular cloud patterns to large-scale subsidence or ascent, Krishnamurthy [5] and Somerville and Gal-Chen [6] discussed the effects of small amounts of throughflow on onset of convection.

Convective instability in a saturated porous medium with throughflow was investigated by Sutton [7], Homsy and Sherwood [8], Jones and Persichetti [9], Nield [10], Khalili and Shivakumara [11], and Shivakumara and Khalili [12]. It is found that in some situation, a small amount of throughflow is destabilizing. While studying the effect of vertical throughflow on the onset of convection in a layer of viscous incompressible fluid between permeable horizontal boundaries heated uniformly from below, Nield [13] found that the effect of throughflow is not invariably stabilizing and depends on the nature of the boundaries specified.

Thermal instability in a horizontal layer of an electrically conducting fluid heated from below and permeated by a uniform vertical magnetic field was investigated by Thompson [14] and Chandrasekhar [15] and experimentally by Nakagawa [16]. It is found that the magnetic field tends to inhibit the onset of thermal convection in the fluid layer and during convection the cells are elongated in the direction of the magnetic lines of force.

The aim of this paper is to study the effects of throughflow on the onset of thermal convection in a horizontal layer of an electrically conducting fluid heated from below in the presence of a uniform vertical magnetic field. The motivation for studying this problem is to explore the possibility of controlling magnetohydrodynamic thermal instability by throughflow.

2. Linear stability analysis

We consider a horizontal layer of an incompressible viscous electrically conducting fluid contained between two rigid permeable plates in the presence of a uniform vertical magnetic field H_0.

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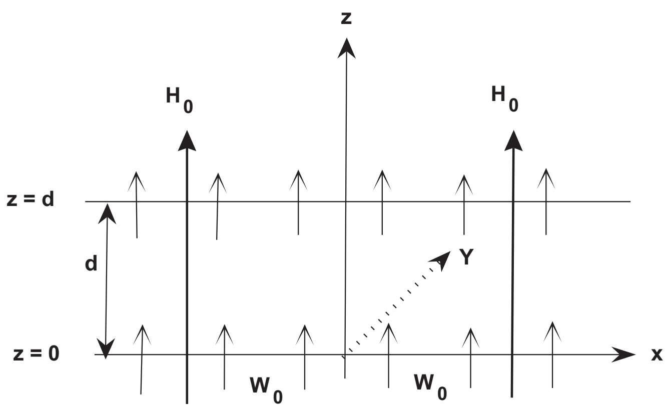

Fig. 1. A sketch of the physical problem.

The layer is heated from below and a constant throughflow of magnitude W_0 is superimposed parallel to the gravity vector. We suppose that the layer is confined by boundaries at $z=0$ and $z=d$ and at both the boundaries, the heat flux is held constant (see Fig. 1).

The basic equations of magnetohydrodynamics (MHD) in the unsteady state are (cf. Chandrasekhar [1])

$$\rho\left(\frac{\partial \mathbf{q}}{\partial t}+\mathbf{q}\cdot\nabla\mathbf{q}\right)=-\nabla\left(p+\frac{\mu_e|\mathbf{H}|^2}{8\pi}\right)+\rho\mathbf{g}+\frac{\mu_e}{4\pi}\mathbf{H}\cdot\nabla\mathbf{H}+\rho\nu\nabla^2\mathbf{q}, \tag{1}$$

$$\nabla\cdot\mathbf{q}=0, \tag{2}$$

$$\nabla\cdot\mathbf{H}=0, \tag{3}$$

$$\frac{\partial \mathbf{H}}{\partial t}+\mathbf{q}\cdot\nabla\mathbf{H}=\mathbf{H}\cdot\nabla\mathbf{q}+\eta\nabla^2\mathbf{H}, \tag{4}$$

where ρ, $\mathbf{q}$, $\mathbf{H}$, p, $\mathbf{g}$, μ_e and ν denote the fluid density, velocity vector, magnetic field vector, pressure, acceleration due to gravity, magnetic permeability and the kinematic viscosity of the fluid, respectively. Further η stands for the magnetic diffusivity of the fluid given by $1/4\pi\mu_e\sigma_e$, where σ_e is the electrical conductivity of the fluid which is assumed constant.

The equation of energy is

$$\frac{\partial T}{\partial t}+\mathbf{q}\cdot\nabla T=\kappa\nabla^2 T+\frac{1}{\rho c_v}\phi+\frac{|\nabla\times\mathbf{H}|^2}{16\pi^2\rho\sigma_e c_v}, \tag{5}$$

where T is the temperature in the fluid, and κ and c_v denote the thermal diffusivity and the specific heat at constant volume of the fluid, respectively. The last term in (5) stands for the Joule heating due to flow of electric current in the fluid and ϕ denotes the viscous dissipation given by

$$\phi=2\mu e_{ij}e_{ij}. \tag{6}$$

Here μ is the dynamic coefficient of viscosity of the fluid and e_{ij} is the rate-of-strain tensor given by (in Cartesian tensor notation)

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right), \tag{7}$$

u_i being the components of velocity $\mathbf{q}$.

The equation of state for the fluid is

$$\rho=\rho_0[1-\alpha(T-T_0)], \tag{8}$$

where α is the coefficient of volume expansion and T_0 is the temperature at which $\rho=\rho_0$. In the unperturbed state, the steady basic temperature distribution $T_{eq}(z)$ is determined from (5) as the solution of

$$W_0\frac{dT_{eq}}{dz}=\kappa\frac{d^2T_{eq}}{dz^2}. \tag{9}$$

This leads to the basic temperature gradient dT_{eq}/dz as

$$\frac{dT_{eq}}{dz}=\frac{W_0\Delta T e^{W_0 z/\kappa}}{\kappa(1-e^{W_0 d/\kappa})}, \tag{10}$$

where ΔT is the imposed temperature difference between the two boundaries. If P stands for the sum of the fluid pressure p and the magnetic pressure $\mu_e|\mathbf{H}|^2/8\pi$, then in the unperturbed state, Eq. (1) gives the equilibrium pressure P_e as the solution of

$$-\frac{dP_e}{dz}-\rho_{eq}g=0. \tag{11}$$

Further from (8)

$$\rho_{eq}=\rho_0[1-\alpha(T_{eq}-T_0)]. \tag{12}$$

The velocity and the magnetic field in the unperturbed state have components $(0,0,W_0)$ and $(0,0,H_0)$, respectively.

Let the initial state described by Eqs. (10)–(12) be slightly perturbed. The perturbed state is assumed as

$$\mathbf{q}=(u,v,W_0+w),\quad P=P_e+P', \tag{13}$$

$$T=T_{eq}(z)+\theta,\quad \mathbf{H}=(h_x,h_y,H_0+h_z), \tag{14}$$

where the perturbation quantities u, v, w, P', θ, h_x, h_y and h_z are functions of x, y, z and t. Substituting (13) and (14) in the momentum equation (1) and using (11) and (12), we get upon ignoring terms of the second and higher orders in the perturbations, the following linearized equations:

$$\frac{\partial u}{\partial t}+W_0\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial P'}{\partial x}+\frac{\mu_e H_0}{4\pi\rho}\frac{\partial h_x}{\partial z}+\nu\nabla^2 u, \tag{15}$$

$$\frac{\partial v}{\partial t}+W_0\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial P'}{\partial y}+\frac{\mu_e H_0}{4\pi\rho}\frac{\partial h_y}{\partial z}+\nu\nabla^2 v, \tag{16}$$

$$\frac{\partial w}{\partial t}+W_0\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial P'}{\partial z}+\frac{\mu_e H_0}{4\pi\rho}\frac{\partial h_z}{\partial z}+g\alpha\theta+\nu\nabla^2 w. \tag{17}$$

In writing the above equations, we have made use of the well known Boussinesq approximation of taking density variation in accounting for the buoyancy force only, the inertial effects of density variation being neglected. The corresponding forms of continuity equation (2) and the solenoidal relation (3) for the magnetic field are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{18}$$

$$\frac{\partial h_x}{\partial x}+\frac{\partial h_y}{\partial y}+\frac{\partial h_z}{\partial z}=0. \tag{19}$$

In addition to the foregoing equations, the linearized forms of the magnetic induction equation (4) and the energy equation (5) are

$$\frac{\partial h_x}{\partial t}+W_0\frac{\partial h_x}{\partial z}=H_0\frac{\partial u}{\partial z}+\eta\nabla^2 h_x, \tag{20}$$

$$\frac{\partial h_y}{\partial t}+W_0\frac{\partial h_y}{\partial z}=H_0\frac{\partial v}{\partial z}+\eta\nabla^2 h_y, \tag{21}$$

$$\frac{\partial h_z}{\partial t}+W_0\frac{\partial h_z}{\partial z}=H_0\frac{\partial w}{\partial z}+\eta\nabla^2 h_z, \tag{22}$$

$$\frac{\partial \theta}{\partial t}+W_0\frac{\partial \theta}{\partial z}+w\frac{dT_{eq}}{dz}=\kappa\nabla^2\theta. \tag{23}$$