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A theoretical study of the first transition for the non-linear Stokes problem in a horizontal annulus

Carlo Ferrario, Arianna Passerini *

Università degli studi di Ferrara, Dipartimento di Matematica e Informatica, Via Machiavelli 30, 44121 - Ferrara, Italy

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ABSTRACT

For any aspect ratio R_o/R_i of the cylinder radii, the non-linear stability of the steady 2D-solutions of the non-linear Stokes system, which is an approximation of the Oberbeck–Boussinesq system, is theoretically studied. The sufficient condition for the stability shows a critical *Ra* which is a function of the aspect ratio. It is the same of the associated homogeneous linear problem and it can be found by looking for the largest eigenvalue of a suitable symmetric operator. The critical *Ra* so defined proves to be uniformly bounded from below in the space of dimensionless parameters, while it is non-uniformly bounded from above for the aspect ratio going to infinity. A scheme to evaluate it as a function of the aspect ratio is given. The results do not depend on the Prandtl number *Pr*.

1. Introduction

Natural convective flows arising in horizontal coaxial cylinders are involved in different applications such as energy storage systems, thermal insulation and cooling. A large amount of experimental data and numerical results, concerning both the flow fields and the heat transfer, can be found in the literature, see for instance [1–6].

In Fig. 1, we show as sketch of the problem a graphical outcome with streamlines and isothermal lines for the steady flows. Here, the temperatures $T_i > T_o$ are fixed at the inner R_i and the outer R_o radii. This geometrical setting gives as immediate result, both experimental and numerical, that natural convective motions are always present for any value, no matter how small, of the Rayleigh number:

$$Ra := \frac{\alpha g}{\nu k}(T_i - T_o)(R_o - R_i)^3$$

(α is the volumetric expansion coefficient, g the gravity acceleration, ν the kinematic viscosity, k the thermal diffusivity).

Actually, all the authors agree on the assertion that for sufficiently small Rayleigh numbers *Ra*, independent of the Prandtl number $Pr := \nu/\kappa$ and of the inverse relative gap width

$$\mathcal{A} = 2R_i/(R_o - R_i),$$

a steady flow with unicellular crescent-shaped eddies occurs.

Although in the papers by Yoo, see Fig. 2, the critical Rayleigh for the first transition seems to be a strongly decreasing function of $\mathcal{A}$, only the basic steady flow is always observed close to the $\mathcal{A}$-axis.

Therefore, in such a region stable steady solutions should exist for the system of partial differential equations which models natural convection. This is actually confirmed by the theoretical papers on the subject [7–11]

All mathematical models in fluid-dynamics are derived from the basic conservation laws, while the Newtonian fluid is the most common model of material [12]. Further, the equations are simplified by means of the Oberbeck–Boussinesq approximation [13,14], whose rigorous derivation under proper hypotheses on the materials is shown in [15] and which is widely studied in several versions depending on the applications (see for instance [16–20]). The four classical assumptions to write the O–B system are:

- isochoric motion: $\nabla \cdot \mathbf{v} = 0$,
- thermal expansion of the material in the weight force: $\rho = \rho_0(1 - \alpha(T - T_0))$,
- uniform density in all the other terms: $\rho = \rho_0$,
- negligible self dissipation: $\mathbb{D} : \mathbb{D} \approx 0$,

(ρ_0 is the reference value for the density and $\mathbb{D}$ is the symmetric part of the gradient of $\mathbf{v}$).

The resulting system is

$$\nabla \cdot \mathbf{v} = 0$$
$$\rho_0\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \mu \Delta \mathbf{v} + \rho_0(1 - \alpha(T - T_0))\mathbf{g}$$
$$\rho_0 C_V\left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\right) = k\Delta T, \quad (1.1)$$

* Corresponding author. Tel.: +39 0532 974825.
E-mail address: arianna.passerini@unife.it (C. Ferrario).

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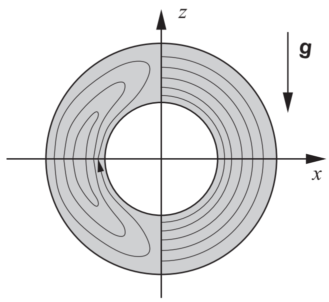

Fig. 1. Basic flow, streamlines (left) and isotherms (right)

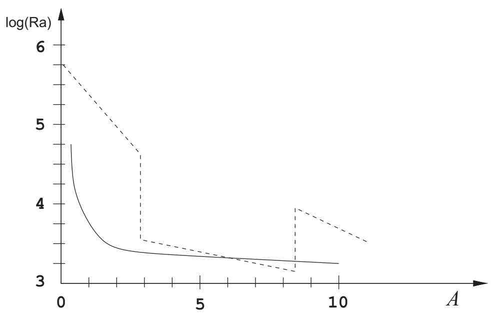

Fig. 2. The line represents the critical Rayleigh number above which dual steady solutions exist. Below the dashed line experiments [2] show prevailing 2d steady flow. The results are for $Pr = 0.7$

where μ is the viscosity and in $\mathbf{g}$ the downward oriented unit vector is included (so that the notation is independent of the frame).

For (approximately) $2.5 < \mathcal{A} < 8.5$, , see Fig. 2, it happens that 3D-flows are observable, say *spiral motions*, while elsewhere in the space of dimensionless parameters only 2D-solutions are observable, so that a 2D-description makes sense. Our 2D-domain is endowed with a reference frame with horizontal x-axis and vertical z-axis, we set $\mathbf{x} = (x, z)$, and the cylindrical axis coincides with the (hidden) y-direction. Of course, polar coordinates (r, φ) with $r \in [R_i, R_o]$ and $\varphi \in [-\pi, \pi)$ come in handy for this geometrical configuration ($\varphi = 0$ corresponds to the positive part of the x-axis). Accordingly, $\mathbf{e}_3 = \mathbf{k} = \nabla(r \sin \varphi) = \sin \varphi \mathbf{e}_r + \cos \varphi \mathbf{e}_\varphi$.

All functions and vector fields are periodic in φ and depend on the dimensionless variable r, in agreement with the following choice:

$$r = \frac{r'}{R_o - R_i}, \quad z = \frac{z'}{R_o - R_i}, \quad t = \frac{\kappa}{(R_o - R_i)^2} t', \quad T = \frac{T'}{T_i - T_o}. \tag{1.2}$$

Next, we write the dimensionless definition of the annulus: for $\mathcal{A} > 0$, the domains in consideration are

$$\Omega_{\mathcal{A}} := \{(r, \varphi) \in \mathbb{R}^2 : r \in (\mathcal{A}/2, 1 + \mathcal{A}/2)\}.$$

If all the variables in the Oberbeck–Boussinesq system are renamed with primes, by expressing all terms as functions of the dimensionless ones, and redefining the pressure by putting together all the gradient-like terms, it follows the system[1]

$$\nabla \cdot \mathbf{v} = 0$$
$$\frac{1}{Pr}\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) - \Delta \mathbf{v} + \nabla \Pi = \frac{Ra}{b} \sin \varphi \mathbf{e}_r + Ra\, \tau \mathbf{e}_3$$
$$\frac{\partial \tau}{\partial t} + \mathbf{v} \cdot \nabla \tau - \Delta \tau = \frac{v^r}{rb}, \tag{1.3}$$

with the boundary conditions

$$\mathbf{v}|_{\partial \Omega_{\mathcal{A}}} = 0, \qquad \tau|_{\partial \Omega_{\mathcal{A}}} = 0,$$

where the third equation follows from the definition of the *excess temperature*:

$$\tau := T - T^* = T - \frac{T_i}{T_i - T_o} + \frac{1}{b}\left(\ln r - \ln \frac{R_i}{R_i - R_o}\right) = T - \frac{T_i}{T_i - T_o} + \frac{1}{b} \ln \frac{2r}{\mathcal{A}}. \tag{1.4}$$

Here, the scalar field T^* is the conductive solution.[2] Moreover, the pressure Π is not the thermodynamical pressure but is redefined by adding gradient-like terms arising from the right hand side of the second equation in (1.1). Of course, $\mathbf{e}_r$ is the unit vector in direction r, $\mathbf{e}_3$ the upward one in direction z, while

$$b := \ln \frac{R_o}{R_i} = \ln\left(1 + \frac{2}{\mathcal{A}}\right), \tag{1.5}$$

is a purely geometric parameter, unbounded as $\mathcal{A}$ tends to zero.

Further, by considering the region in which stable basic steady flows occur, one can see how the linear Stokes-like system, studied in [3], works as approximation of the complete model which is most commonly used: it gives a stable flow for any Ra close to zero. However, in [8] it is proved that by using such a simplified model, although the fluid flows for any Ra, the heat transfer is the same as for conduction: the Nusselt number is not increased by the motion if one neglects the transport term in the heat equation. This feature makes the model very weak in describing how the thermal energy can be transported by convection.

The present paper deals with the non-linear stability of the non-linear Stokes problem [21], a less simplified and more realistic approximation of the full system of equations. This simplified version of (1.3) is got by erasing only the non-linear term $\mathbf{v} \cdot \nabla \mathbf{v}$ lying in the linear momentum balance.

The mathematical tools herein used are the classical ones of the functional analysis applied to fluid dynamics and can be found, for instance in [22–24].

In particular, in the next section by using the Straughan scheme [25], we define the critical Rayleigh number Ra_{cr} of the non-linear Stokes problem in the annulus. It depends on the curvature of the domain but it is uniformly bounded from below, and we prove that for $Ra < Ra_{cr}(\mathcal{A})$ the steady solutions, which can easily be found with the same techniques as in [11], are asymptotically stable and then unique. Actually, a further bound $Ra < Ra_*(\mathcal{A})$ must hold, confining Ra to a region of the space of parameters which is however unbounded.

In Section 3, we furnish the computational scheme to identify $Ra_{cr}(\mathcal{A})$, hoping that such a function, once plotted by numerical methods, will be compared with Yoo's first transition line.

2. The non-linear stability

As announced in the Introduction, the model in consideration is

$$\nabla \cdot \mathbf{v} = 0$$
$$\frac{1}{Pr}\frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v} + \nabla \Pi = \frac{Ra}{b} \sin \varphi \mathbf{e}_r + Ra\, \tau \mathbf{e}_3$$

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[1] Though this is seldom specified, in polar coordinates the symbol $\mathbf{v} \cdot \nabla \mathbf{v}$ is a strong abuse of notation, since the sum over repeated indices $v^i \partial_i \mathbf{v}$ is understood in Cartesian coordinates. In polar coordinates, $\mathbf{v} \cdot \nabla \mathbf{v}$ actually stands for

$$\mathbf{v} \cdot \nabla \mathbf{v} = \begin{pmatrix} \partial_r v^r & \frac{1}{r}(\partial_\varphi v^r - v^\varphi) \\ \partial_r v^\varphi & \frac{1}{r}(v^r + \partial_\varphi v^\varphi) \end{pmatrix} \begin{pmatrix} v^r \\ v^\varphi \end{pmatrix} = \begin{pmatrix} v^r \partial_r v^r + \frac{v^\varphi}{r}(\partial_\varphi v^r - v^\varphi) \\ v^r \partial_r v^\varphi + \frac{v^\varphi}{r}(v^r + \partial_\varphi v^\varphi) \end{pmatrix}.$$

In what follows, this involved expression is understood unless its manipulation was strictly necessary.

[2] In particular, T^* solves $\Delta T^* = 0$ with boundary conditions $T^*(\mathcal{A}/2, \varphi) = T_i/(T_i - T_o)$ and $T^*(\mathcal{A}/2 + 1, \varphi) = T_o/(T_i - T_o)$. So that, with the same boundary conditions, T solves $-\Delta T + \mathbf{v} \cdot \nabla T = 0$.