International Journal of Non-Linear Mechanics 78 (2016) 9–16

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Invariant linearization criteria for a three-dimensional dynamical system of second-order ordinary differential equations and applications

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ARTICLE INFO

Article history:
Received 16 May 2015
Received in revised form
20 September 2015
Accepted 21 September 2015
Available online 30 September 2015

Keywords:
Invariant
Linearization
Free particle system
Dynamical system

ABSTRACT

Second-order dynamical systems are of paramount importance as they arise in mechanics and many applications. It is essential to have workable explicit criteria in terms of the coefficients of the equations to effect reduction and solutions for such types of equations. One important aspect is linearization by invertible point transformations which enables one to reduce a non-linear system to a linear system. The solution of the linear system allows one to solve the non-linear system by use of the inverse of the point transformation. It was proved that the n-dimensional system of second-order ordinary differential equations obtained by projecting down the system of geodesics of a flat $(n+1)$-dimensional space can be converted to linear form by a point transformation. This is a generalization of the Lie linearization criteria for a scalar second-order equation. In this case it is of the maximally symmetric class for a system and the linearizing transformation as well as the solution can be directly written down. This was explicitly used for two-dimensional dynamical systems. The criteria were written down in terms of the coefficients and the linearizing transformation allowed for the general solution of the original system. Here the work is extended to a three-dimensional dynamical system and we find explicit criteria, including the linearization test given in terms of the coefficients of the cubic in the first derivatives of the system and the construction of the transformations, that result in linearization. Applications to equations of classical mechanics and relativity are given to illustrate our results.

1. Introduction

Lie symmetry analysis can be used to find reductions and exact solutions of linear and non-linear (systems of) differential equations (DEs) that are invariant under a sufficient number of point transformations, i.e., transformations of the independent and dependent variables [1] that leave invariant the DEs. However, the procedures although algorithmic can be quite cumbersome. The problem reduces enormously if the transformations can convert the DEs to simpler form. The simplest form is the linear form. For higher-order DEs it may be useful to utilize generalized or other higher transformations which are more general than point transformations. For a scalar second-order ODE, Lie [2] showed that the necessary and sufficient conditions for linearization by point transformation are that the ODE be at most cubic in the first derivative and that its coefficients satisfy an over-determined system of four conditions involving their partial derivatives along with those of two auxiliary functions. Lie obtained, by invertible change of variables, both practical (in terms of the coefficients) and algebraic criteria for a scalar second-order ODE to be reducible to the simple free particle equation. Tressé [3] derived two invariants of the equivalence group of point transformations for a scalar second-order ODE and proved that their vanishing provides the necessary and sufficient conditions for its linearization to the free particle equation. These conditions have been proved to be equivalent to the Lie linearization conditions [4]. They were derived in [5] by the use of geometric arguments and for the Cartan equivalence method in [6]. Important applications of symmetries and reductions to ODEs are to Emden-type and reduced non-Newtonian equations. Such types of ODEs have received much attention [7–9]. Another interesting area is that of symmetries of first integrals of second-order ODEs. It has been

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http://dx.doi.org/10.1016/j.ijnonlinmec.2015.09.016
0020-7462/

shown that if a first integral of a scalar second-order ODE has three symmetries then it is linearizable by point transformation [10]. Algebraic linearization criteria via an invertible change of variables for a dynamical system of second-order ODEs have been obtained in [11,12] and the complete symmetry classification of a linear system of two second-order ODEs was investigated in [13,14]. Recently, Bagderina [15] found linearization criteria for a dynamical system of two second-order ODEs. A review of linearization of ODEs by various methods and some generalizations is given in [16]. Since every system of ODEs can be transformed to a (larger) system of first order ODEs by introducing new variables corresponding to the derivatives of the higher order equation, it might have been expected that the linearization problem reduces to one of analyzing first order linearization, as has been done for control problems, for example in [17]. By setting the control to zero one would obtain the system of first order ODEs. In fact, it is well-known that the use of Lie point symmetries can convert every system of first order ODEs to linear form [1]. However, there is no canonical procedure for determining the Lie symmetries of first order systems and one loses the power of Lie symmetry analysis for this case [2]. Further, the Lie symmetries (and hence all methods using them) do not carry through under the conversion to first-order systems. As such, the linearizability of first-order ODEs says nothing about that of higher-order ODEs. That is why Lie needed to develop his analysis in the first place. As has been shown [18] with the 2-dimensional systems that are maximally symmetric, one gets to use the power of geometry to effectively just write down the solution of the system using algebraic computing.

The set of symmetries of the system of geodesic equations properly contains the set of isometries of the underlying manifold [20,21]. This leads naturally to linearization criteria for a dynamical system of second-order ODEs that have the same form as geodesic equations [18]. Following the projection procedure of Aminova and Aminov [22] and by utilization of the above connection, linearization criteria for a system of cubically semi-linear second-order ODEs to the simplest free particle system have been provided in [23]. In that work 2-dimensional systems were investigated in some detail. By projecting the geodesic equations in 2-dimensional to a scalar ODE, the Lie linearization conditions were automatically derived. Using it projecting 3-dimensional systems of geodesic equations, one could directly write down the linearizing transformations and the solution for the most symmetric class of linearizable 2-dimensional systems. While the method was stated in full generality, its implementation in three dimensions needed to be explicitly studied. This is performed in this paper. Thus we investigate the linearizability of a system of three second-order cubically semi-linear equations to say the free particle system. We derive the linearization criteria to the simplest system in terms of the coefficients of the equations. We also show how one can construct the linearizing point transformation in a systematic manner. Moreover, we show that they can be used for the reduction of the linear system of three equations to the simplest free particle equations.

The plan of the paper is as follows. A brief review of the mathematical notation is given in the next section. Next practical criteria in terms of the coefficients for linearization of three cubically semi-linear second-order equations to the simplest system are given. We also show how the point transformations can be constructed. This transformation hen allows one to recover the solution of the original system. Herein we state the relevant results for a linear system as well. To illustrate our results we present applications, to classical mechanics and relativity, in the subsequent section. A brief summary and conclusion are given in the last section.

2. Mathematical notation

For completeness, it is crucial to first present some geometrical notation (see, e.g. [20,22]). The Christoffel symbols, in terms of the metric tensor g_{ij}, are given as

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{im}(g_{jm,k} + g_{km,j} - g_{jk,m}), \tag{1}$$

where g^{im} is the inverse metric tensor, i.e., $g^{im}g_{jm} = \delta^i_j$. These Christoffel symbols have $n^2(n+1)/2$ coefficients and are symmetric in the lower two indices. The system of geodesic equations is

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0, \quad i,j,k = 1,\ldots,n, \tag{2}$$

where the dot denotes the total differentiation with respect to the parameter s. The Riemann curvature tensor is

$$R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}, \tag{3}$$

which is skew-symmetric in the lower last two indices and satisfies

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0. \tag{4}$$

A necessary and sufficient condition for a system of n quadratically non-linear second-order ODEs of the form (2) to be linearizable by point transformation is that the Riemann tensor vanishes, i.e.,

$$R^i_{jkl} = 0. \tag{5}$$

Following Aminova and Aminov [22], we project the system of geodesic equations (2) down by one dimension and write the system of second order ODEs as

$$x^{a''} + A_{bc}x^{a'}x^{b'}x^{c'} + B^a_{bc}x^{b'}x^{c'} + C^a_b x^{b'} + D^a = 0, \quad a = 2,\ldots,n, \tag{6}$$

where the prime now represents differentiation with respect to the parameter x^1 and the coefficients in terms of the Γ^a_{bc} are

$$\begin{aligned} &A_{bc} = -\Gamma^1_{bc}, \quad B^a_{bc} = \Gamma^a_{bc} - 2\delta^a_{(c}\Gamma^1_{b)1}, \quad C^a_b = 2\Gamma^a_{1b} - \delta^a_b\Gamma^1_{11}, \\ &D^a = \Gamma^a_{11}, \quad a,b,c = 2,\ldots,n, \end{aligned} \tag{7}$$

in which we have used the notation $T_{(ab)} = (T_{ab} + T_{ba})/2$.

3. Linearization conditions for cubically semi-linear equations

We are motivated by the success in obtaining the Lie conditions for a scalar second-order cubically semi-linear ODE and the derivation of the practical criteria for linearizing a system of two second-order semi-linear ODEs derived in [23]. We pursue similar conditions for linearization of a system of three second-order ODEs. Consequently, we study (6) when $n=3$ for linearization via point transformations by resorting to a system of four geodesic equations (2). Here we consider practical linearization criteria for reduction to the simplest system in terms of the coefficients for a system of three cubically semi-linear second-order ODEs of the form (6), $n=3$.

For $n=4$, Eqs. (6) and (7) are written as

$$\begin{aligned} &x^{2''} + A_{22}(x^{2'})^3 + 2A_{23}(x^{2'})^2x^{3'} + 2A_{24}(x^2)'^2x^{4'} + A_{33}x^{2'}(x^{3'})^2 \\ &\quad + 2A_{34}x^{2'}x^{3'}x^{4'} + A_{44}x^{2'}(x^{4'})^2 + B^2_{22}(x^{2'})^2 + 2B^2_{23}x^{2'}x^{3'} + 2B^2_{24}x^{2'}x^{4'} \\ &\quad + B^2_{33}(x^{3'})^2 + 2B^2_{34}x^{3'}x^{4'} + B^2_{44}(x^{4'})^2 + C^2_2x^{2'} + C^2_3x^{3'} + C^2_4x^{4'} + D^2 = 0, \end{aligned}$$

$$\begin{aligned} &x^{3''} + A_{22}(x^{2'})^2x^{3'} + 2A_{23}x^{2'}(x^{3'})^2 + 2A_{24}x^{2'}x^{3'}x^{4'} + A_{33}(x^{3'})^3 \\ &\quad + 2A_{34}(x^{3'})^2x^{4'} + A_{44}x^{3'}(x^{4'})^2 + B^3_{22}(x^{2'})^2 + 2B^3_{23}x^{2'}x^{3'} + 2B^3_{24}x^{2'}x^{4'} \\ &\quad + B^3_{33}(x^{3'})^2 + 2B^3_{34}x^{3'}x^{4'} + B^3_{44}(x^{4'})^2 + C^3_2x^{2'} + C^3_3x^{3'} + C^3_4x^{4'} + D^3 = 0, \end{aligned}$$

$$\begin{aligned} &x^{4''} + A_{22}(x^{2'})^2x^{4'} + 2A_{23}x^{2'}x^{3'}x^{4'} + 2A_{24}x^{2'}(x^{4'})^2 + A_{33}(x^{3'})^2x^{4'} \\ &\quad + 2A_{34}x^{3'}(x^{4'})^2 + A_{44}(x^{4'})^3 + B^4_{22}(x^{2'})^2 + 2B^4_{23}x^{2'}x^{3'} + 2B^4_{24}x^{2'}x^{4'} \end{aligned}$$