International Journal of Non-Linear Mechanics 78 (2016) 45–52

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Large deflection and post-buckling analysis of non-linearly elastic rods by wavelet method

Lei Zhang, Jizeng Wang*, You-He Zhou

Key Laboratory of Mechanics on Disaster and Environment in Western China, Ministry of Education, College of Civil Engineering and Mechanics, Lanzhou University, Lanzhou, Gansu 730000, China

ARTICLE INFO

Article history:
Received 1 February 2015
Received in revised form
27 August 2015
Accepted 7 October 2015
Available online 17 October 2015

Keywords:
Wavelet Galerkin method
Large deflection
Post-buckling analysis
Non-linearly elastic rod
Ludwick constitutive law

ABSTRACT

We propose a wavelet method in the present study to analyze the large deflection bending and post-buckling problems of rods composed of non-linearly elastic materials, which are governed by a class of strong non-linear differential equations. This wavelet method is established based on a modified wavelet approximation of an interval bounded $\mathbf{L}^2$-function, which provides a new method for the large deflection bending and post-buckling problems of engineering structures. As an example, in this study, we considered the rod structures of non-linear materials that obey the Ludwick and the modified Ludwick constitutive laws. The numerical results for both large deflection bending and post-buckling problems are presented, illustrating the convergence and accuracy of the wavelet method. For the former, the wavelet solutions are more accurate than the finite element method and the shooting method embedded with the Euler method. For the latter, both bifurcation and limit loads can be easily and directly obtained by solving the extended systems. On the other hand, for the shooting method embedded with Runge–Kutta method, to obtain these values usually needs to choose a good starting value and repeat trial solutions many times, which can be a tough task.

1. Introduction

Lightweight structures such as rods, plates, and shells can usually undergo a large deflection within the range of small strains. If such structural elements comprise some type of materials that obey non-linearly elastic constitutive laws, both geometrical non-linearity and material non-linearity must be considered in the analysis of the mechanical behavior of these elements, which is a basic characterization of many engineering systems, such as flocked fabrics, robotic arms, bridge and engine mountings, and structural damps [1,2]. Many polymeric fiber and metallic fiber materials exhibit a non-linear constitutive relationship [2]. Experiments have also shown that cantilever beams made of a common type of stainless steel exhibit highly non-linear stress–strain curves [3]. Structural elements made of such materials have attracted the attention of many researchers since the 1970s, such as Antman and Rosenfeld have provided an analytical analysis of global buckling of rods with non-linear constitutive laws [4]. Although the work of Antman and Rosenfeld have not provided any results of the specific material laws and respective physical behavior of structures, their work demonstrates the significance of material non-linearity in buckling problems [5]. Bars are found to have the tendency to buckle well before reaching the bifurcation buckling load when a specific material non-linearity is considered in buckling analysis [6–9]. The critical force that induces the buckling of rods is called the limit load. Beams made of non-linearly elastic materials also exhibit quite a rich non-linear bending behavior [1,10].

The governing equation of these non-linearly elastic rods can usually be reduced to a quasi-linear differential equation (DE), which reflects the non-linearity of the constitutive equations. A quasi-linear DE is non-linear in (at least) one of the lower derivatives but is linear in the highest order derivative of an unknown function [11], which is a class of strong non-linear DEs. In the present study, we consider the non-linear Ludwick constitutive law, and the modified variation of this law in the bending and buckling analysis of rods. For bending problems of rods with the Ludwick constitutive law, an analytical solution in the form of a definite integral [12] and a semi-analytical solution in terms of the tangent of a bending angle [13] have been provided. However, the analytical solution ceases to exist when the bending angle reaches a critical value, and the semi-analytical solution is difficult to determine except for a few cases. An approximate formula to determine the limit load has been obtained for buckling analysis, but the application range of this formula is minimal [14]. Therefore, developing numerical methods for these problems is essential to investigate their rich non-linear behaviors. The governing

* Corresponding author.
E-mail address: jzwang@lzu.edu.cn (J. Wang).

http://dx.doi.org/10.1016/j.ijnonlinmec.2015.10.002
0020-7462/

Table 1
Coefficients p_k.

k	p_k	k	p_k	k	p_k
0	−2.392638657280051E−03	6	6.459945432939942E−01	12	1.238869565706006E−02
1	−4.932601854180402E−03	7	1.116266213257999E+00	13	−1.583178039255944E−02
2	2.714039971139949E−02	8	5.381890557079980E−01	14	−2.717178600539990E−03
3	3.064755594619984E−02	9	−9.961543386239989E−02	15	2.886948664020020E−03
4	−1.393102370707997E−01	10	−7.992313943479994E−02	16	6.304993947079994E−04
5	−8.060653071779983E−02	11	5.149146293240031E−02	17	−3.058339735960013E−04

equation, which is a second order non-linear DE with two-point boundary conditions, is usually solved through the shooting method. The basic idea of the shooting method in solving boundary value problems is to rewrite the boundary value problems into initial value problems by adding an extra boundary condition at the start point. The initial value problems can then be solved through various kinds of numerical integration methods, such as the Runge–Kutta method [2,10,14–16] and the Euler method [17]. However, the shooting method requires repeating trials in practical computation to determine an appropriate value for the extra boundary condition that satisfies the boundary conditions at another point. Moreover, care must be taken in choosing a good starting value not only for singular point itself but also for its corresponding eignfunction when solving the buckling problems via shooting method [18]. In the present study, we propose a wavelet-based Galerkin method for solving this type of non-linear differential equation.

Wavelets are a newly developed powerful mathematical tool, which shows the potential in numerical analysis. Wavelets provide another way to decompose/construct a function space compared with the conventional Fourier theory [19]. Based on wavelet theory, we have developed several efficient numerical methods for various problems [20–24]. Very few efficient wavelet methods have been developed for the buckling problems of light weight structures. A wavelet collocation method is applied for the buckling analysis of laminated plates [25] and a wavelet finite element method (FEM) is employed for plates based on Reissner–Mindlin theory [26]. However, these analyzes are limited to linear problems without considering geometrical non-linearity and material non-linearity. Based on a simple and accurate wavelet approximation expansion for any interval bounded square integrable function [27], we propose a wavelet Galerkin method for bending and buckling problems of rods with non-linear constitutive laws. The governing equation of these problems is then discretizing into a set of non-linear algebraic equations. The Jacobian matrix of these non-linear algebraic equations is singular when load reaches the limit point or bifurcation point. Here, we adopt the extended system method to determine the limit and bifurcation loads. In contrast to the commonly used incremental method, the extended system method does not require tracing post-buckling paths, and these singular points can be obtained directly [28,29]. The numerical results of both bending and buckling problems are presented in this paper, which demonstrate the convergence and accuracy of the wavelet Galerkin method.

2. Wavelet Galerkin method

The governing equation for the large deflection bending and post-buckling problems of a wide class of non-linearly elastic rods can be given by [6–8,10]:

$$\frac{d^2\theta}{ds^2} = \mathbf{N}(s, \theta, \kappa) \tag{1}$$

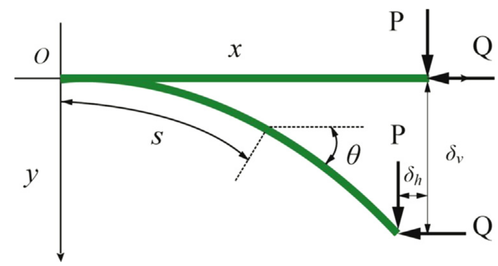

Fig. 1. Configuration of the rod subjected to a combined load.

where $\theta(s)$ and $\kappa(s) = d\theta/ds$ are the bending angle and curvature of the rod, **N** is a non-linear operator, respectively. Boundary conditions are at the ends of a rod assumed to be Dirichlet or Neumann conditions.

Eq. (1) is a strong non-linear differential equation when the rod is under large deflection both in bending and buckling problems. The wavelet Galerkin method for solving this type of equations is established based on a modified wavelet approximation of an interval bounded $\mathbf{L}^2$-function.

2.1. A brief introduction to wavelet

According to the multiresolution analysis of wavelet theory [30], the function space $\mathbf{L}^2(\mathrm{R})$ can be divided to a sequence of nested subspaces $\{0\}\cdots \subset \mathrm{V}_0 \subset \mathrm{V}_1 \subset \cdots \subset \mathrm{V}_j \subset \mathrm{V}_{j+1} \subset \cdots \subset \mathbf{L}^2(\mathrm{R})$. A set of orthogonal basis of subspace V_j can be formed by

$$\phi_{j,k}(x) = 2^{\frac{j}{2}}\phi(2^j x - k), k \in \mathbf{Z} \tag{2}$$

and a function $f(x) \in \mathbf{L}^2(R)$can be approximated through projecting this function from $\mathbf{L}^2(\mathrm{R})$ to V_j as

$$f(x) \approx \mathbf{P}^j f(x) = \sum_k c_{j,k}\phi_{j,k}(x) \tag{3}$$

where $\phi(x)$is the orthogonal scaling function, and $c_{j,k} = \int_{-\infty}^{\infty} f(x)\phi_{j,k}(x)dx$. Scaling function with compact support can be constructed by using low pass filter coefficients p_k with a finite number in terms of the relation below

$$\phi(x) = \sum_k p_k \phi(2x - k) \tag{4}$$

in which subscript $k = 0,1,2,\ldots,3N-1$ for Coiflet-type wavelet, $N-1$ is the number of vanishing moment of the corresponding wavelet function [31]. Such a scaling function has the unique property of shifted vanishing moments

$$\int_{-\infty}^{\infty} (t - M_1)^k \phi(t) dt = 0, 1 \le k < N \tag{5}$$

where $M_1 = \int_{-\infty}^{\infty} x\phi(x)dx$. Based on this unique property, one has $c_{j,k} \approx 2^{-j/2} f(\frac{k+M_1}{2^j})$, so that approximation of the function can be