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Non-linear evolution of a sinusoidal pulse under a Brinkman-based poroacoustic model

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ABSTRACT

Through numerical analyses, we study the roles of Brinkman viscosity, the Darcy coefficient, and the coefficient of non-linearity on the evolution of finite amplitude harmonic waves. An investigation of acoustic blow-ups is conducted, showing that an increase in the magnitude of the non-linear term gives rise to blow-ups, while an increase in the strength of the Darcy and/or Brinkman terms mitigate them. Finally, an analytical study via a regular perturbation expansion is given to support the numerical results.

1. Introduction

The study of poroacoustics, the phenomenon in which an acoustic propagation in a viscous fluid is obtained within a porous medium [1], has been of great interest to numerous fields of science and industry, namely oil exploration, medical ultrasound testing, acoustic insulation, and the food industry. Straughan [2] provides more details on these and other examples.

It is generally understood that Darcy's Law governs poroacoustic propagation [3]:

$$\nabla \mathcal{P} = -\left(\frac{\mu\chi}{K}\right)\mathbf{V}, \tag{1}$$

where $\mathcal{P}$ is the intrinsic pressure, μ is the dynamic viscosity, χ is the porosity, K is the permeability, and $\mathbf{V}$ is the intrinsic velocity. This expression models the behavior of the acoustic potential when considering the fluid–pore interactions alone. However, Payne et al. [4] argue that if a boundary or interface is present, or if the porosity is near unity, i.e. the fluid–pore interaction is not the dominating factor, then Brinkman's equation should be used [3]:

$$\nabla \mathcal{P} = \tilde{\mu}\chi\nabla^2\mathbf{V} - \left(\frac{\mu\chi}{K}\right)\mathbf{V}. \tag{2}$$

Here, $\tilde{\mu}$ is the Brinkman or effective viscosity. This expression not only accounts for the fluid–pore interaction found in the Darcy expression, it also models the fluid–fluid interactions. In this work we will use a finite-difference scheme to investigate the time evolution of an acoustic wave within a finite boundary.

2. Mathematical formulation and problem statement

As in [5], we obtain the basic governing equation for a weakly non-linear model based on (2). Assuming 1D flow, conservation of mass, conservation of momentum, and the quadratic approximation to the isentropic equation of state [6] take the form:

$$\varrho_t + (\varrho u)_x = 0, \tag{3}$$

$$\varrho(u_t + uu_x) = -\mathcal{P}_x + \tilde{\mu}\chi u_{xx} - \left(\frac{\mu\chi}{K}\right)u, \tag{4}$$

$$\mathcal{P} = \mathcal{P}_e + \varrho_e c_e^2[s + (\beta-1)s^2], \tag{5}$$

where ϱ is the mass density and u is the velocity. Here and henceforth, we will use the notation $\eta_i := \partial\eta/\partial i$, where η is a general variable. Note that the subscript e denotes the (constant) equilibrium state, c is the adiabatic speed of sound, β is the coefficient of non-linearity, and $s = (\varrho - \varrho_e)/\varrho_e$ is the condensation.

From these equations, eliminating $\mathcal{P}$ from (4) using (5) we get

$$\varrho(u_t + uu_x) = -\varrho_e c_e^2[s + (\beta-1)s^2]_x + \tilde{\mu}\chi u_{xx} - \left(\frac{\mu\chi}{K}\right)u. \tag{6}$$

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We now introduce the following dimensionless quantities, noting that $u=\phi_x$, where $\phi=\phi\ (x,t)$ is the velocity potential:

$$\phi^\diamond=\frac{\phi}{VL},\quad u^\diamond=\frac{u}{V},\quad x^\diamond=\frac{x}{L},\quad t^\diamond=\frac{tc_e}{L},\quad P^\diamond=\frac{\mathcal{P}-\mathcal{P}_e}{\varrho_e c_e V}. \tag{7}$$

Here, V and L are the characteristic speed and length, both positive, respectively. Also, by introducing the Mach number $\epsilon=V/c_e$, and after further manipulation as well as using the binomial expansion approximation, dropping all terms of order $\mathcal{O}(\epsilon^2)$, we obtain [1]

$$\Box^2\phi+\chi(Re)^{-1}\phi_{txx}-\delta\phi_t=\epsilon\partial_t[(\beta-1)\phi_t^2+\phi_x^2], \tag{8}$$

where $\Box^2\equiv\partial_{xx}-\partial_{tt}$ is the 1D d'Alembertian operator, $Re=c_e L\sigma_e/\mu$ is a Reynolds number, $\delta\propto\chi$ is the dimensionless Darcy coefficient, and we have dropped the diamond superscripts on the dimensionless quantities. As in [7], we will now use the Lighthill–Westervelt approximation:

$$\phi_x\approx-\phi_t, \tag{9}$$

on the right hand side of (8). We next differentiate both sides of (8) with respect to t, and use the approximation:

$$P\approx-\phi_t, \tag{10}$$

which follows from (the dimensionless) Bernoulli's equation. This substitution, along with the following boundary and initial conditions, gives us the initial boundary-value problem (IBVP):

$$\Box^2P+\chi(Re)^{-1}P_{txx}-\delta P_t=-2\epsilon\beta[(P_t)^2+P(P_{tt})], \tag{11a}$$

$$P(x,0)=\sin(\pi x),\quad P_t(x,0)=0\ \text{for}\ (0<x<1), \tag{11b}$$

$$P(0,t)=0,\quad P(1,t)=0\ \text{for}\ (t>0). \tag{11c}$$

3. Numerical analysis

3.1. Finite difference scheme construction

With the IBVP having been stated, we will now make use of a finite difference scheme to find a numerical solution, meaning (11) must be discretized. The first step is to select the integers $I\geq 2$ and $J\geq 2$. Next, we set $\Delta x=T/I$ and $\Delta t=T/J$, where Δx and Δt denote uniform spatial and temporal step sizes, and T is the value of t for which the solution is sought. This gives the mesh points $x_i=i(\Delta x)$, for all $i=0,1,\ldots,I$, and $t_j=i(\Delta t)$, for all $j=0,1,\ldots,J$.

With this done, we will discretize (11a). We will start by replacing the second order derivatives with centered difference quotients, and first order derivatives with a backwards-Euler quotient. With these replacements, we obtain the difference equation,

$$\frac{P_{i+1}^j-2P_i^j+P_{i-1}^j}{\Delta x^2}-\frac{P_i^{j+1}-2P_i^j+P_i^{j-1}}{\Delta t^2}+\frac{\chi}{Re}\left[\frac{(P_{i+1}^j-2P_i^j+P_{i-1}^j)-(P_{i+1}^{j-1}-2P_i^{j-1}+P_{i-1}^{j-1})}{\Delta x^2\Delta t}\right]-\delta\left(\frac{P_i^j-P_i^{j-1}}{\Delta t}\right)$$
$$=-2\epsilon\beta\left[\left(\frac{P_i^j-P_i^{j-1}}{\Delta t}\right)^2+P_i^j\left(\frac{P_i^{j+1}-2P_i^j+P_i^{j-1}}{\Delta t^2}\right)\right], \tag{12}$$

where $P_i^j\approx P(x_i,t_j)$. Solving for the most advanced time step P_i^{j+1}, we obtain, after some manipulation, the explicit scheme:

$$P_i^{j+1}=\Delta t\frac{\chi}{Re}\left(\frac{P_{i-1}^{j-1}-P_{i-1}^j-2P_i^{j-1}+2P_i^j+P_{i+1}^{j-1}-P_{i+1}^j}{\Delta x^2(2\epsilon\beta P_i^j-1)}\right)+\Delta t^2\left(\frac{-P_{i-1}^j+2P_i^j-P_{i+1}^j}{\Delta x^2(2\epsilon\beta P_i^j-1)}\right)+\Delta t\delta\left(\frac{-P_i^{j-1}+P_i^j}{2\epsilon\beta P_i^j-1}\right)$$
$$+\left(\frac{P_i^{j-1}-2\delta P_i^j+2\epsilon\beta((P_i^j)^2+P_i^{j-1}P_i^j-(P_i^{j-1})^2)}{2\epsilon\beta P_i^j-1}\right). \tag{13}$$

The last step involves discretizing the initial and boundary conditions (11b) and (11c). For details on this straightforward process see [8]. The resulting conditions are as follows:

$$P_i^0=\sin(\pi x_i), \tag{14a}$$

$$\partial_t P_i^0=0, \tag{14b}$$

$$P_0^j=0, \tag{14c}$$

$$P_I^j=0. \tag{14d}$$

3.2. Numerical results

Fig. 1 is a plot of the pressure, P, as a function of position, vs time. It was plotted with high Darcy coefficient and very high Reynolds number, of the order of 20,000. We have assumed a Mach number of 0.01 and a value of the non-linearity coefficient $\beta=3.625$. This corresponds to seawater at 20 °C and 3.5% salinity [9]. These values would diminish the Brinkman term enough to force the system to correspond to the Darcy–Jordan Model (DJM) [1]. We see that the system does indeed evolve like a damped oscillator.

Fig. 2 shows the corresponding amplitude of the pressure as a function of position at various time snapshots.

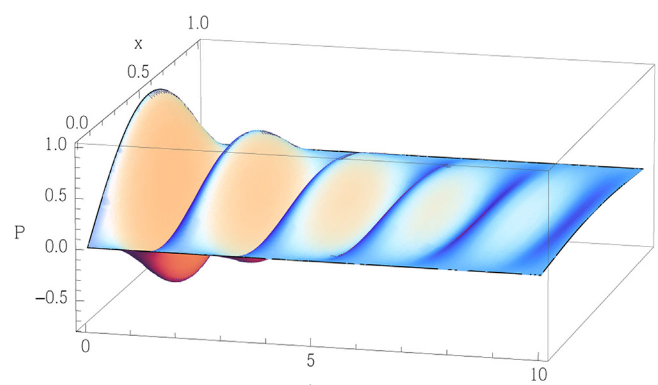

Fig. 1. P vs x vs t for $\epsilon=0.01$, $\delta=0.5$, $\chi=.9$, $\beta=3.625$, and $Re=20{,}000$.

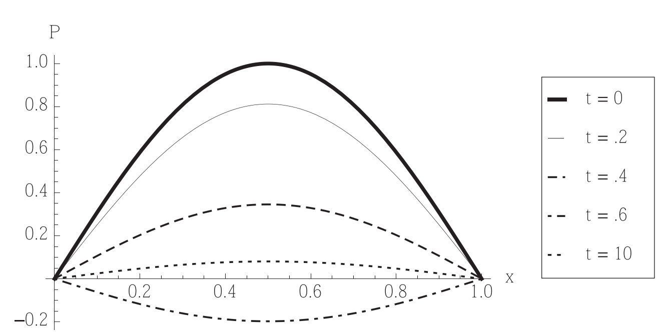

Fig. 2. P vs x for $\epsilon=0.01$, $\delta=0.5$, $\chi=0.9$, $\beta=3.625$, and $Re=20{,}000$ at varying times.