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Squeeze flow of a Bingham-type fluid with elastic core

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ABSTRACT

The squeeze flow of a Bingham-type material between finite circular disks is considered. The material is modelled assuming that the unyielded region behaves like a linear elastic core. A lubrication approximation is considered. It is shown that no paradox can arise, such as that has been pointed out for many years by various authors when the unyielded region in the fluid is supposed to be perfectly rigid. The unyielded region is shown to be always detached from the axis of symmetry. Some numerical simulations are worked out for different squeezing rates.

1. Introduction

The squeeze flow between circular discs is often encountered in many devices used to determine the flow properties of highly "viscous" materials such as concrete, molten polymers, ceramic pastes etc. Most of these materials are constitutively modeled as Bingham plastics [1], that is continua associated to a "plastic" criterion (Von Mises) that links the stress state and the yield stress. Where the criterion is locally satisfied, a velocity gradient arises in the medium and the body starts to flow as a linear viscous fluid. If the criterion is not fulfilled, there is no velocity gradient and the material is stationary or moves as a rigid plug [2]. The yield surface separating the yielded and unyielded regions is generally unknown.

In many situations the geometrical setting of the problem is such that the aspect ratio is negligibly small, so that lubrication approximation can be used, see [3]. While, on the one hand, lubrication allows for major simplifications of the governing equations, on the other, it may cause the emergence of paradoxes and inconsistencies that invalidate the main constitutive assumptions. As stated by Covey and Stanmore [3] and subsequently by Wilson in [4], there is an immediate difficulty when one deals with Bingham squeeze lubrication flow as the expected yield surface clashes with the model. Indeed, simple symmetry arguments require that the shear stress (which is dominant) decreases below the yield stress close to the mid-plane. Hence, the flow criterion is not fulfilled there and an unyielded region forms around the mid-plane. Because of the cylindrical geometry, the plug has to be stationary, but, at the same time, the gap between the plates is being narrowed and the plug has to deform. The solution thus becomes inconsistent and a paradox arises.

This result, that was first pointed out by Lipscomb and Denn in [5], led the authors to argue that a true rigid plug cannot exist in complex geometries, since the lubrication scaling predicts unyielded plugs that move with a velocity that slowly varies in the principal flow direction (the so called pseudo-plugs, see [6,7]). Though the paradox remains true for axisymmetric squeeze flow, in many other complex geometries asymptotic solutions that predict truly rigid regions have been found. Balmforth and Craster [8] have proposed a procedure that allows to construct consistent solutions for thin-layer problems, a technique subsequently exploited by Frigaard and Ryan [9] for the Bingham flow in a channel of slowly varying width.

In the last decades many ways of overcoming the "lubrication paradox" paradox have been developed. For example Gartling and Phan-Thien [10] and Wilson [4] have proposed to substitute the original Bingham model with a bi-viscous model in which the solid behaviour is never required even for zero shear rate. Others [11,12] have used exponential viscosity models or power-law fluid models [13,14]. We refer the readers to [15–18,11], where a vast literature on such an issue can be found. In particular an exhaustive review of the regularization models and their implementation can be found in [19], while an excellent review on yield stress fluids can be found in [20]. Recently Fusi et al. [21] have proposed a new procedure where the rigid plug is treated as an evolving non-material volume and where the momentum balance of the unyielded region is written through an integral formulation. This procedure allows to determine a true plug at the leading order of the lubrication approximation with no need to define a pseudo-plug or a fake yield surface. The same approach has been used in Fusi et al. [22] to study the squeeze flow of a Bingham fluid in planar geometry. In the paper by Muravleva [23] the planar squeeze flow is studied following the technique introduced in

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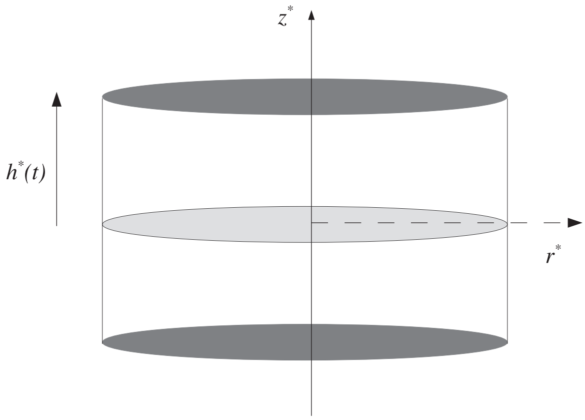

Fig. 1. A schematic representation of the system.

Balmforth and Craster [8] and Frigaard and Ryan [9]. In the specific case of axisymmetric squeeze flow Smyrnaios and Tsamopoulos [11] have shown that unyielded material may exist only around stagnation points located at the center of the disks and their results are confirmed by numerical simulations.

In this paper, following an approach developed in [24,25], we consider the axisymmetric squeeze flow of a Bingham-like material and we overcome the lubrication paradox by modelling the material as a Bingham fluid with a deformable core. In particular we model the unyielded domain as a linear elastic solid, but other constitutive relations could be used. By doing this we allow the solid plug to deform and no paradox arises at the mid-plane placed between the plates. In practice we are considering a yield stress fluid which behaves like a linear elastic solid when the stress is below a fixed threshold. We remark that the idea of modelling the core as an elastic material dates back to the works by Oldroyd [2], and by Yoshimura and Prud'homme [26]. However, to our knowledge, such a model was never applied to the squeezing between circular discs.

We consider the continuum confined between two parallel discs, both of radius[1] R^*, moving one toward the other in a prescribed way thus causing the squeezing of the fluid (Fig. 1). Denoting by $h^*(t^*)$ the half distance between discs, we assume

$$\varepsilon=\frac{R^*}{H^*}\ll 1,\quad H^*=\sup_{t^*\geq 0} h^*(t^*),\quad \frac{dh^*}{dt^*}\leq 0, \tag{1.1}$$

so that the lubrication approximation is justified. We assume that the material behaves as a linear viscous fluid if the stress is above a critical threshold τ_o^* and as a linear elastic solid when stress state is below τ_o^*. We develop the mathematical model at the leading order and show that the model predicts the existence of an evolving yield surface which, however, appears only after some time depending on the squeezing velocity. After the emergence of the yield surface the domain is split in two regions: (i) an elastic domain where the material is unyielded; (ii) a sheared domain where the critical stress is overcome.

We will show that, at the leading order of the lubrication approximation, the region around the axis of symmetry remains always unyielded, so that the viscous region is always detached from that axis. We will perform some numerical simulation for different given expressions of the squeezing rate plotting the evolution of the yield surface and of the pressure profile.

[1] Throughout the paper starred quantities indicate dimensional quantities.

2. The Mathematical model

Let us consider a mechanically incompressible continuum occupying a domain like the one depicted in Fig. 1. Suppose that two circular plates of radius R^* are squeezing the fluid confined in between and are approaching each other with prescribed motion $\pm h^*(t^*)$, with $\pm h^*(0)=\pm H^*$. As a consequence, the material is squeezed out radially. In radial polar coordinates[2] the displacement is given by

$$\mathbf{u}^*=u_r^*(r^*,z^*,t^*)\mathbf{e}_r+u_z^*(r^*,z^*,t^*)\mathbf{e}_r, \tag{2.1}$$

while velocity is expressed by

$$\mathbf{v}^*=v_r^*(r^*,z^*,t^*)\mathbf{e}_r+v_z^*(r^*,z^*,t^*)\mathbf{e}_r, \tag{2.2}$$

We also assume that the displacement is *small* (infinitesimal strain theory), so that the Lagrangian description and the Eulerian description are essentially the same.

Being $\mathbf{T}^*=-p^*\mathbf{I}+\mathbf{S}^*$ the Cauchy stress tensor, $p^*=(1/3)\mathrm{tr}\,\mathbf{T}^*$ the pressure, as in [24] we make the following constitutive assumption for the deviatoric part of $\mathbf{T}^*$:

$$\begin{cases} II_S^*<\tau_o^*, & \mathbf{S}^*=\left(2\eta^*+\frac{\tau_o^*}{II_D^*}\right)\mathbf{D}^*, & \text{viscous model,}\\ II_S^*<\tau_o^*, & \mathbf{S}^*=2k^*\mathbf{E}, & \text{linear elastic model,}\\ II_S^*=\tau_o^*, & & \text{yield condition,}\end{cases} \tag{2.3}$$

where

$$\mathbf{D}^*=\tfrac{1}{2}\left[\overset{*}{\nabla}\mathbf{v}^*+(\overset{*}{\nabla}\mathbf{v}^*)^T\right],\quad \mathbf{E}=\tfrac{1}{2}\left[\overset{*}{\nabla}\mathbf{u}^*+(\overset{*}{\nabla}\mathbf{u}^*)^T\right],$$

- $II_S^*=\left[\frac{1}{2}\mathbf{S}^*\cdot\mathbf{S}^*\right]^{1/2}$ and $II_D^*=\left[\frac{1}{2}\mathbf{S}^*\cdot\mathbf{S}^*\right]^{1/2}$ are, respectively, the second invariant of $\mathbf{S}^*$ and $\mathbf{D}^*$,
- η^* is the viscosity of the fluid, τ_o^* the stress yield threshold and k^* the elastic modulus of the solid phase.

In practice the constitutive model (2.3) represents a Bingham-like material which behaves as a linear elastic body when $II_S^*<\tau_o^*$, and as a linear viscous fluid as $II_S^*>\tau_o^*$.

Mass balance is given by

$$\frac{\partial v_z^*}{\partial z^*}+\frac{\partial v_r^*}{\partial r^*}+\frac{v_r^*}{r^*}=0, \tag{2.4}$$

while momentum conservation in the absence of body force is expressed by

$$\begin{cases} \varrho\left(\frac{\partial v_r^*}{\partial t^*}+v_r^*\frac{\partial v_r^*}{\partial r^*}+v_z^*\frac{\partial v_r^*}{\partial z^*}\right)=-\frac{\partial p^*}{\partial r^*}+\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*S_{rr}^*)+\frac{\partial S_{rz}^*}{\partial z^*}-\frac{S_{\theta\theta}^*}{r^*},\\ \varrho\left(\frac{\partial v_z^*}{\partial t^*}+v_r^*\frac{\partial v_z^*}{\partial r^*}+v_z^*\frac{\partial v_z^*}{\partial z^*}\right)=-\frac{\partial p^*}{\partial z^*}+\frac{1}{r^*}\frac{\partial}{\partial r^*}(r^*S_{rz}^*)+\frac{\partial S_{zz}^*}{\partial z^*}.\end{cases} \tag{2.5}$$

The second invariant of $\mathbf{S}^*$ writes

$$II_S^*=\left[\frac{1}{2}\mathbf{S}^*\cdot\mathbf{S}^*\right]^{1/2}=\sqrt{\left[S_{rz}^{*2}+\frac{1}{2}\left(S_{rr}^{*2}+S_{\theta\theta}^{*2}+S_{zz}^{*2}\right)\right]}. \tag{2.6}$$

The non-zero components of $\mathbf{D}^*$ and $\mathbf{E}^*$ are

$$D_{rr}^*=\left(\frac{\partial v_r^*}{\partial r^*}\right),\quad D_{zz}^*=\left(\frac{\partial v_z^*}{\partial z^*}\right),\quad D_{\theta\theta}^*=\left(\frac{v_r^*}{r^*}\right)\quad D_{rz}^*=\frac{1}{2}\left(\frac{\partial v_r^*}{\partial z^*}+\frac{\partial v_z^*}{\partial r^*}\right),$$

$$E_{rr}=\left(\frac{\partial u_r^*}{\partial r^*}\right),\quad E_{zz}=\left(\frac{\partial u_z^*}{\partial z^*}\right),\quad E_{\theta\theta}=\left(\frac{u_r^*}{r^*}\right)\quad E_{rz}=\frac{1}{2}\left(\frac{\partial u_r^*}{\partial z^*}+\frac{\partial u_z^*}{\partial r^*}\right),$$

whereas the second invariant of $\mathbf{D}^*$ is more conveniently written

[2] We assume no deformation/velocity in the θ-direction, as well radial symmetry, that is all relevant variables are independent of the polar coordinate θ.