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Dynamics of an oscillator with delay parametric excitation

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ABSTRACT

This paper involves the dynamics of a delay limit cycle oscillator being driven by a time-varying perturbation in the delay:

$$\dot{x} = -x(t-T(t)) - \epsilon x^3$$

with delay $T(t) = \frac{\pi}{2} + \epsilon k + \epsilon \cos \omega t$. This delay is chosen to periodically cross the stability boundary for the $x=0$ equilibrium in the constant-delay system.

For most of parameter space, the system is non-resonant, leading to quasiperiodic behavior. However, a region of 2:1 resonance is shown to exist where the system's response frequency is entrained to half of the forcing frequency ω. By a combination of analytical and numerical methods, we find that the transition between quasiperiodic and entrained behavior consists of a variety of local and global bifurcations, with corresponding regions of multiple stable and unstable steady-states.

1. Introduction

A recent study [1] of dynamical systems with delayed terms has considered the following "delay limit cycle oscillator" in the form of a differential-delay equation (DDE):

$$\dot{x} = -x(t-T_0) - \epsilon x^3 \tag{1}$$

This system exhibits a supercritical Hopf bifurcation at delay $T_0 = \pi/2$ such that the equilibrium point at the origin $x=0$ is stable for $T_0 < \pi/2$ and unstable otherwise. The stable limit cycle for $T_0 > \pi/2$ is created with natural frequency 1 [2–4]. For an introduction to DDEs, see [5].

Eq. (1) with $\epsilon = 0$ has had application to insect locomotion [6].

This paper considers a system of the same form as Eq. (1), but with a periodically time-varying delay $T(t) = \pi/2 + \epsilon k + \epsilon \cos \omega t$:

$$\dot{x} = -x(t-T(t)) - \epsilon x^3 = -x\left(t - \frac{\pi}{2} - \epsilon k - \epsilon \cos \omega t\right) - \epsilon x^3 \tag{2}$$

The delay T is taken to be time-dependent such that the system may periodically cross the Hopf bifurcation exhibited by the constant T case. This causes the stability of the $x=0$ equilibrium to regularly alternate between stable and unstable. We would anticipate the equilibrium being stable if it is in the stable region for more than half of the forcing period, and unstable otherwise. However, we will show that the effect of this forcing may cause unexpected behavior due to resonance between the forcing frequency ω and the frequency of the limit cycle created in the Hopf.

The effect of time-periodic delay on an oscillator has been studied with application to turning processes with varying spindle speed in machine-cutting [7].

2. Non-resonant two-variable expansion

We begin by expanding the system about the $\epsilon = 0$ solution, using two time variables, fast time u and slow time v:

$$u = t \quad v = \epsilon t \quad x = x_0 + \epsilon x_1 + O(\epsilon^2) \tag{3}$$

The multiple time scales lead to the restatement of the derivative:

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt} = x_u + \epsilon x_v \tag{4}$$

The delay term is also approximated by its Taylor expansion:

$$x_d = x(u-T, v-\epsilon T) = x\left(u - \frac{\pi}{2}, v\right) - \epsilon(k + \cos \omega u) x_u\left(u - \frac{\pi}{2}, v\right) - \epsilon \frac{\pi}{2} x_v\left(u - \frac{\pi}{2}, v\right) + O(\epsilon^2) \tag{5}$$

Within the original equation with these expansions applied, we can find the coefficients of each power of ϵ. The O (1) terms ($\epsilon = 0$) give the differential equation:

$$x_{0u} + x_0\left(u - \frac{\pi}{2}, v\right) = 0 \tag{6}$$

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with a solution of the form:

$$x_0(u,v)=A(v)\ \cos\ u+B(v)\ \sin\ u \tag{7}$$

The $O(\epsilon)$ terms are found from the original (expanded) equation to give an equation for x_1:

$$x_{1u}+x_1\left(u-\frac{\pi}{2},v\right)=-x_{0v}+(k+\ \cos\ \omega u)x_{0u}\left(u-\frac{\pi}{2},v\right)+\frac{\pi}{2}x_{0v}\left(u-\frac{\pi}{2},v\right)-x_0^3 \tag{8}$$

Since we will be looking to eliminate secular terms $\cos(u)$ and $\sin(u)$, at this point we note that some terms' resonance or non-resonance are dependent on the value of ω, in particular:

$$(\cos\ \omega u)(A\ \cos\ u+B\ \sin\ u)=\frac{A}{2}(\cos(\omega+1)u+\cos(\omega-1)u)+\frac{B}{2}(\sin(\omega+1)u - \sin(\omega-1)u) \tag{9}$$

Here we will split our analysis into two cases, resonant ($\omega\approx 2$) and non-resonant ($\omega\not\approx 2$), in order to account for the presence or absence of the resonant terms that arise from $\cos\ \omega u$.

2.1. Non-resonant behavior

Eliminating secular terms in the case where $\omega\not\approx 2$ results in the approximation and slow flow:

$$(2\pi^2+8)A'=8kA-4\pi kB+(3\pi B-6A)(A^2+B^2) \tag{10}$$

$$(2\pi^2+8)B'=8kB+4\pi kA-(3\pi A+6B)(A^2+B^2) \tag{11}$$

Transforming to polar form by taking $A=R\ \cos\ \theta$ and $B=R\ \sin\ \theta$, to result in the new form $x_0(u,v)=R(v)\cos(u-\theta(v))$, gives:

$$(\pi^2+4)R'=R(4k-3R^2) \tag{12}$$

$$(2\pi^2+8)\theta'=\pi(4k-3R^2) \tag{13}$$

The R' equation is uncoupled, allowing us to study it separately. $R=0$ solves $R'=0$ for all parameter values (representing the origin $x=0$); this solution is stable for all $k<0$. For $k>0$ the stable solution is $R=4k/3$, with a corresponding $\theta'=0$. Based on this result, $x(t)$ is approximated to have response frequency 1 for $k>0$, as in the original limit cycle oscillator Eq. (1) with $T_0>\pi/2$.

We note that in this expansion, the periodic forcing is shown to have no effect to this order. By expanding about the resonant forcing frequency $\omega=2$ below, we will see that the second frequency does have an effect on the non-resonant behavior as well as behavior within the resonant region.

3. Resonant two-variable expansion

According to Eq. (9), the choice $\omega=2$ makes the system resonant. To consider this behavior, we will redefine the two-variable expansion about this value by defining $\omega=2+\epsilon\Delta$.

Using new variables for fast time ξ and slow time η to expand about the resonance at $\omega=2$:

$$\omega t=2\xi=2(1+\epsilon\Delta/2)t \quad \eta=\epsilon t \tag{14}$$

$$\dot{x}=\frac{dx}{dt}=\frac{\partial x}{\partial\xi}\frac{d\xi}{dt}+\frac{\partial x}{\partial\eta}\frac{d\eta}{dt}=(1+\epsilon\Delta/2)x_\xi+\epsilon x_\eta \tag{15}$$

We proceed as before. The x_0 solution takes the same form as Eq. (7) in terms of the new time variables:

$$x_0(\xi,\eta)=A(\eta)\cos\ \xi+B(\eta)\sin\ \xi \tag{16}$$

while the $O(\epsilon)$ terms give the following equation for x_1:

$$x_{1\xi}+x_1\left(\xi-\frac{\pi}{2},\eta\right)=-x_{0\eta}-\frac{\Delta}{2}x_{0\xi}+\left(\frac{\pi\Delta}{4}+k+\ \cos\ 2\xi\right)x_{0\xi}\left(\xi-\frac{\pi}{2},\eta\right)+\frac{\pi}{2}x_{0\eta}\left(\xi-\frac{\pi}{2},\eta\right)-x_0^3 \tag{17}$$

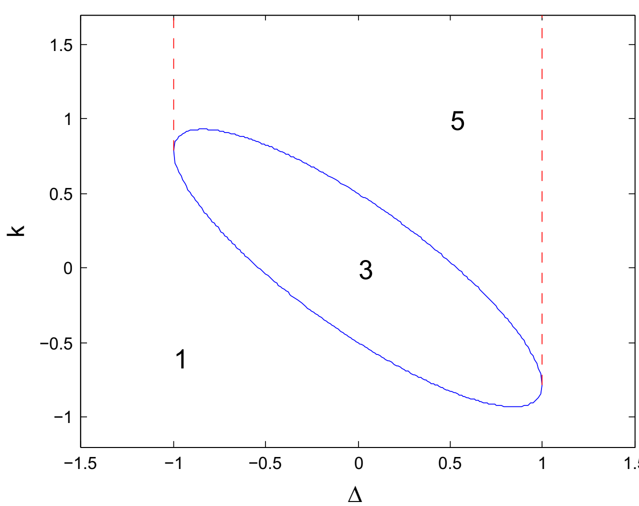

Fig. 1. Regions with 1, 3, and 5 slow flow equilibria, bounded by (dashed) double saddle-node bifurcations and (solid) pitchfork bifurcations.

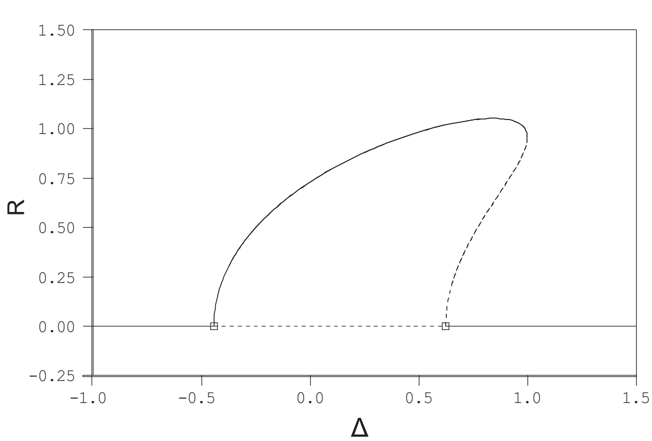

Fig. 2. AUTO results for $k=-0.1$ with varying Δ. Plotting the amplitude $R=\sqrt{A^2+B^2}$ of the $x(t)$ response for the equilibria, with stability information (solid is stable, dashed is unstable). All points on the $R\neq 0$ curve represent 2 equilibria by symmetry.

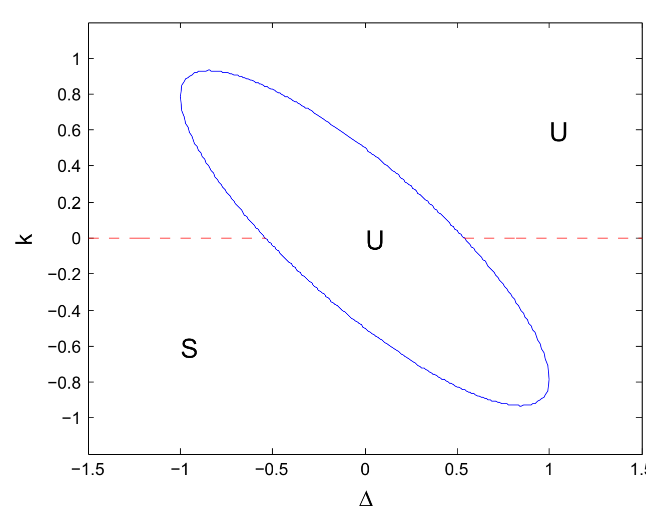

Fig. 3. Stability of $x=0$ near the resonance; “U” is unstable, “S” is stable. Changes in stability are caused by pitchfork bifurcations (solid line) and Hopf bifurcations (dashed line).