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Fractional Birkhoffian method for equilibrium stability of dynamical systems

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ABSTRACT

In this paper, we present a new method, i.e. fractional Birkhoffian method, for stability of equilibrium positions of dynamical systems, in terms of Riesz derivatives, and study its applications. For an actual dynamical system, the fractional Birkhoffian method of constructing a fractional dynamical model is given, and then the seven criterions for fractional Birkhoffian method of equilibrium stability are established. As applications, by using the fractional Birkhoffian method, we construct four kinds of actual fractional dynamical models, which include a fractional Duffing oscillator model, a fractional Whittaker model, a fractional Emden model and a fractional Hojman–Urrutia model, and we explore the equilibrium stability of these models respectively. This work provides a general method for studying the equilibrium stability of an actual fractional dynamical system that is related to science and engineering.

1. Introduction

With the development of science and technology, the Hamiltonian dynamics has played an important role not only in modern mechanics, but also in non-linear science, engineering science, physics, mathematics and so on [1–7]. In 1927, Birkhoff gave an extension of Hamiltonian system and presented a new integral variational principle and a new form of the equations of motion in his famous works [8]. In 1978, Santilli carried out primary research on the Birkhoffian system and gained some results. In addition to the generalization of Galilei's relativity, Santilli studied the Birkhoffian equations, the transformation theory of Birkhoffian equations and so on [9,10]. Since then, Mei and his co-workers constructed the theoretical framework of Birkhoffian mechanics [11–13]. The Birkhoffian mechanics is more general than the Hamiltonian mechanics. The Hamiltonian mechanics has been extensively applied in many fields of science and engineering, so the Birkhoffian mechanics should also play a more important role in these fields, and have been applied to non-linear mechanics [14–16], relativistic mechanics [17], rotational relativistic mechanics [18] and quantum mechanics [19], etc. The stability of dynamical systems is an important problem both in theories and applications, and has attracted many researchers [20–26]. However, the fractional Birkhoffian method for equilibrium stability of a dynamical system is not presented so far.

Fractional dynamical method not only can more truly reveal the natural phenomena, but also is more close to the engineering practice. In the end of the 1970s, Mandelbrot discovered a fact that a large number of fractional dimension examples existed in nature and engineering, and fractional calculus became a useful and important tool for researching fractal geometry [27]. This important discovery caused the shock of science, and scientists began to study many problems about the dynamical system with fractional derivatives. In 1996, Riewe established the fractional Lagrangian formulation and Hamiltonian formulation of classical mechanics for non-conservative systems, and his two papers are the starting point that the fractional calculus was introduced to classical mechanics [28,29]. Basing on different definitions of fractional derivatives, Agrawal [30–33], Baleanu [33–36], Klimek [37,38] and Cresson [39] respectively researched fractional Euler–Lagrange variation problems, and established Lagrangian equations and Hamiltonian equations with different fractional derivatives. Baleanu [40,41] further explored fractional Hamiltonian systems, which include fractional Hamiltonian analysis of the $1+1$ dimensional field theory, the fractional Ostrogradski's formulation and the irregular system with linearly dependent constraints. Tarasov [42,43] presented the fractional generalization of non-holonomic constraints defined by fractional derivatives, and given the Lagrangian formulation and Hamiltonian formulation with fractional non-holonomic constraints. Their work [28–43] constructs the basic theories and methods for fractional Lagrangian mechanics and fractional Hamiltonian mechanics. Further, Luo and his co-workers established fractional generalized Hamiltonian

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mechanics, which include fractional generalized Hamiltonian equations, gradient representation, Lie algebraic structure, Poisson conservation law, variation equations, construction method of integral invariants and stability of the manifold of equilibrium states [44–47], and presented the fractional Nambu dynamics [48]. The study of the basic theories and methods for fractional dynamics has become a hot topic, and been extensively applied in various fields of engineering, physics, mechanics, biology, nonlinear science and so on [43,49–60]. Recently, we established fractional Birkhoffian mechanics, which include unified fractional Pfaff–Birkhoff principle, fractional Pfaff–Birkhoff–D'Alembert principle, fractional Birkhoffian equations and its tensor representation, and so on [61,62]. In order to better solve the fractional dynamical stability problems in science and engineering, it is necessary to propose the fractional Birkhoffian method for equilibrium stability of dynamical systems.

In the paper, we present a new method, i.e., fractional Birkhoffian method, for stability of equilibrium positions of dynamical systems, and study its applications.

Section 2 explains briefly the fractional Birkhoffian system, and provides a fractional Birkhoffian method of constructing fractional dynamical models.

In Section 3, we present the fractional Birkhoffian method of equilibrium stability, and give its seven criterions.

Section 4 discusses special cases.

In Applications A–D of Sections 5–8, by using the fractional Birkhoffian method, we construct four kinds of fractional dynamical models, which include a fractional Duffing oscillator model, a fractional Whittaker model, a fractional Emden model and a fractional Hojman–Urrutia model, and we explore the stability of equilibrium positions of these models, respectively.

Section 9 contains the conclusions.

2. Fractional Birkhoffian method of constructing fractional dynamical model

Let us consider a fractional Birkhoffian system of which the local coordinates of a mechanical system are determined by Birkhoffian variable $a^{\nu}(\nu = 1, 2, \ldots, 2n)$. Birkhoffian is $B(t, a)$, Birkhoff functions are $R_{\nu}(t, a)$, fractional derivative ${}^{R}_{t_1}D^{\alpha}_{t_2}a$ is n times continuously differentiable in interval $[t_1, t_2]$, n is a positive integer, and $0 < \alpha < 1$. If the differential equations of motion of a fractional mechanical system can be expressed in the following forms [61]:

$$\frac{\partial R_{\nu}(t,a)}{\partial a^{\mu}}\,{}^{R}_{t_1}D^{\alpha}_{t_2}a^{\nu} - {}^{RC}_{t_1}D^{\alpha}_{t_2}R_{\mu}(t,a) - \frac{\partial B(t,a)}{\partial a^{\mu}} = 0, \quad (\mu, \nu = 1, 2, \ldots, 2n), \tag{1}$$

then the system is called the fractional Birkhoffian system with Riesz derivatives. If Birkhoff functions R_{ν} do not include time t, Eq. (1) is reduced to the autonomous fractional Birkhoffian system

$$\frac{\partial R_{\nu}(a)}{\partial a^{\mu}}\,{}^{R}_{t_1}D^{\alpha}_{t_2}a^{\nu} - {}^{RC}_{t_1}D^{\alpha}_{t_2}R_{\mu}(a) - \frac{\partial B(t,a)}{\partial a^{\mu}} = 0, \quad (\mu, \nu = 1, 2, \ldots, 2n). \tag{2}$$

The tensor representation of Eq. (2) can be written as

$$\omega_{\mu\nu}\,{}^{R}_{t_1}D^{\alpha}_{t_2}a^{\nu} - \frac{\partial B(t,a)}{\partial a^{\mu}} = 0, \quad (\mu, \nu = 1, 2, \ldots, 2n), \tag{3}$$

where

$$\omega_{\mu\nu} = \frac{\partial R_{\nu}(a)}{\partial a^{\mu}} - \frac{\partial R_{\mu}(a)}{\partial a^{\nu}} \tag{4}$$

is the Birkhoff tensor. Eq. (3) can also be written as

$${}^{R}_{t_1}D^{\alpha}_{t_2}a^{\mu} = \omega^{\mu\nu}\frac{\partial B(t,a)}{\partial a^{\nu}}, \quad (\mu, \nu = 1, 2, \ldots, 2n), \tag{5}$$

if and only if $\det(\omega_{\mu\nu}) \neq 0$. Here

$$\omega^{\mu\nu} = \left(\|\omega_{\chi\delta}\|^{-1}\right)^{\mu\nu} \tag{6}$$

is called fractional Birkhoff contravariant tensor.

For an actual dynamical system, if we can construct its Birkhoffian B and Birkhoff functions R_{ν}, then by using fractional Birkhoffian equations, the fractional dynamical model of this system can be established. This method is called fractional Birkhoffian method of constructing fractional dynamical models.

3. Fractional Birkhoffian method for equilibrium stability of a dynamical system

We know that for an actual dynamical system, if the $2n$ equilibrium equations are independent of each other, then these equilibrium positions are isolated. In this section, the fractional Birkhoffian method of equilibrium stability is investigated, and the seven criterions are given.

Suppose the equilibrium positions of autonomous fractional Birkhoffian Eq. (5) are

$${}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\mu} = \left({}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\mu}\right)^{0} = const, \quad {}^{R}_{t_1}D^{\alpha}_{t_2}a^{\mu} = 0, \quad (\mu = 1, 2, \ldots, 2n). \tag{7}$$

Substituting Eq. (7) into Eq. (5), we obtain the equilibrium equations

$$\left[\omega^{\mu\nu}\frac{\partial B(a)}{\partial a^{\nu}}\right]_0 = 0, \quad (\mu, \nu = 1, 2, \ldots, 2n), \tag{8}$$

so that

$$\left[\frac{\partial B(a)}{\partial a^{\nu}}\right]_0 = 0, \quad (\nu = 1, 2, \ldots, 2n), \tag{9}$$

where $()_0$ means that $\left({}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\mu}\right)^{0}$ takes the place of ${}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\mu}$. The system has equilibrium positions (7) when Eq. (9) has solutions.

Suppose that equilibrium positions (7) are isolated. If the equilibrium positions given by Eq. (7) are disturbed by the action of small disturbance ξ^{μ}, then we have

$${}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\mu} = \left({}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\mu}\right)^{0} + {}^{R}_{t_1}D^{\alpha-1}_{t_2}\xi^{\mu}, (\mu = 1, 2, \ldots, 2n), \tag{10}$$

and substituting Eq. (10) into Eq. (5), we obtain perturbation equations

$${}^{R}_{t_1}D^{\alpha}_{t_2}\xi^{\mu} = (\omega^{\mu\nu})_1\left[\frac{\partial B(a)}{\partial a^{\nu}}\right]_1, \quad (\mu, \nu = 1, 2, \ldots, 2n), \tag{11}$$

where $()_1$ means that $\left({}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\mu}\right)^{0} + {}^{R}_{t_1}D^{\alpha-1}_{t_2}\xi^{\mu}$ takes the place of ${}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\mu}$. Expanding $(\omega^{\mu\nu})_1$ and $(\frac{\partial B}{\partial a^{\nu}})_1$ into Taylor series near equilibrium positions, we have

$$(\omega^{\mu\nu})_1 = (\omega^{\mu\nu})_0 + \sum_{\rho=1}^{2n}\left(\frac{\partial \omega^{\mu\nu}}{\partial\,{}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\rho}}\right)_0 {}^{R}_{t_1}D^{\alpha-1}_{t_2}\xi^{\rho} + \ldots, \tag{12}$$

$$\left(\frac{\partial B}{\partial a^{\nu}}\right)_1 = \left(\frac{\partial B}{\partial a^{\nu}}\right)_0 + \sum_{\rho=1}^{2n}\left(\frac{\partial^2 B}{\partial a^{\nu}\partial\,{}^{R}_{t_1}D^{\alpha-1}_{t_2}a^{\rho}}\right)_0 {}^{R}_{t_1}D^{\alpha-1}_{t_2}\xi^{\rho} + \ldots. \tag{13}$$

Substituting Eqs. (12) and (13) into Eq. (11), and using Eqs. (8) and (9), we can obtain the perturbation equations

$${}^{R}_{t_1}D^{\alpha}_{t_2}\xi^{\mu} = A\,{}^{R}_{t_1}D^{\alpha-1}_{t_2}\xi^{\rho} + \Lambda_{\mu}(t, {}^{R}_{t_1}D^{\alpha-1}_{t_2}\xi^{\mu}), \quad (\mu, \rho = 1, 2, \ldots, 2n), \tag{14}$$