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Steady and oscillatory convection in rotating fluid layers heated and salted from below

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ABSTRACT

Double convection in rotating horizontal layers filled by a Navier–Stokes fluid mixture, heated and salted from below, is investigated. Onset of linear instability – for any value of the fluid and salt Prandtl numbers P_r, P_1 – either via the Routh–Hurwitz conditions or via steady or oscillatory convection, is characterized. Introducing a new field connecting the perturbation fields to the temperature and salt concentration, in the cases $P_1 = 1$ or $P_r = 1$ or $P_1P_r = 1$, stability conditions in algebraic closed form are obtained. Linear stability is recovered as non-linear global asymptotic stability via the Auxiliary System Method.

1. Introduction

Because of the numerous applications in the real world phenomena (industrial processes, water pollution, geology, volcanism, etc.), many efforts have been devoted to analyze the stability of the thermal conduction solution of multicomponent Navier–Stokes fluid mixture in the absence of rotation [3,4,12–14,17,18] and in the more realistic case of the presence of rotation [1,2,5–7, 9–11,15,16,19,20]. However, as far as we know, in the case of double convection in rotating layers either the onset of linear instability or the non-linear energy stability analysis is not completely investigated. In fact – as concerns the linear instability – the onset of convection via steady or oscillatory state is not completely characterized via algebraic closed form. Further – as concerns the non-linear energy stability – the coincidence between the linear and non-linear stability conditions is obtained generally under severe restrictions on the initial data. In the present paper we reconsider the problem in the case of rotating layers heated and salted from below, aimed to characterize via algebraic closed forms the onset of instability via steady or oscillatory convection. In particular, our scope is to show that: (1) in the cases $P_1 = 1$ or $P_r = 1$ or $P_1P_r = 1$, the onset of convection can be characterized via algebraic closed form by introducing a new unknown field; (2) linear stability can be recovered as non-linear global asymptotic stability via the Auxiliary System Method (see [13–17]).

2. Mathematical model

Let $\mathbb{L}$ be a horizontal layer of depth d filled by a Navier–Stokes fluid mixture in which a chemical specie (salt) S is dissolved in and let $Oxyz$ be an orthogonal frame of reference with fundamental unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards). We suppose that $\mathbb{L}$ is uniformly heated from below and rotates uniformly about the vertical axis with constant angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$. The equat ions governing the fluid motion, in the Boussinesq approximation, are [1,7]:

$$\begin{cases} \rho_0(\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v}) = -\nabla P + \rho_0\nu\Delta\mathbf{v} - 2\rho_0\omega\mathbf{k}\times\mathbf{v} - \rho_0[1 - A(T - T_0) + A_1(C - C_0)]g\mathbf{k}, \\ \nabla\cdot\mathbf{v} = 0, \\ T_t + \mathbf{v}\cdot\nabla T = k\Delta T, \\ C_t + \mathbf{v}\cdot\nabla C = k_1\Delta C, \end{cases} \tag{2.1}$$

with ρ_0 being constant density, $P = p - \frac{\rho_0}{2}|\boldsymbol{\omega}\times\mathbf{x}|^2$, $\mathbf{x} = (x, y, z)$, $\mathbf{v}$ the fluid velocity, T the temperature, C the salt concentration, p the pressure, T_0 the reference temperature, $\mathbf{g} = -g\mathbf{k}$ the gravity, C_0 the reference salt concentration, ν the kinematic viscosity, A the thermal expansion coefficient, A_1 the salt expansion coefficient, k the thermal diffusivity, k_1 the salt diffusivity.

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To (2.1) we append the boundary conditions

$$\begin{cases} T(x,y,0,t)=T_l, \quad T(x,y,d,t)=T_u, \quad T_l>T_u \\ C(x,y,0,t)=C_l, \quad C(x,y,d,t)=C_u, \quad C_l>C_u \\ \mathbf{v}\cdot\mathbf{k}=0, \quad \text{on } z=0,d. \end{cases} \tag{2.2}$$

The boundary value problem (2.1)–(2.2) admits the thermal conduction solution $\overline{m}_0=(\overline{p},\overline{\mathbf{v}},\overline{T},\overline{C})$ given by

$$\begin{cases} \overline{\mathbf{v}}=\mathbf{0}, \quad \overline{T}=T_l-\dfrac{\delta T}{d}z, \quad \overline{C}=C_l-\dfrac{\delta C}{d}z, \\ \delta T=T_l-T_u, \quad \delta C=C_l-C_u, \\ \overline{p}(z)=\overline{p}_0-\rho_0 gz[1-A(T_l-T_0)+A_1(C_l-C_0)]-\dfrac{\rho_0 gz^2}{2d}[A\delta T-A_1\delta C]+\dfrac{\rho_0\omega^2 z^2}{2}, \quad \overline{p}_0=\text{const.}>0. \end{cases} \tag{2.3}$$

Setting

$$p=\overline{p}+\pi, \quad \mathbf{v}=\overline{\mathbf{v}}+\mathbf{u}, \quad T=\overline{T}+\theta, \quad C=\overline{C}+\Phi \tag{2.4}$$

and introducing the non-dimensional scalings

$$\begin{cases} t=t^*\dfrac{d^2}{k}, \quad \mathbf{u}=\mathbf{u}^*\dfrac{\nu}{d}, \quad \pi=\pi^*\dfrac{\nu^2\rho_0}{d^2}, \\ \mathbf{x}=\mathbf{x}^* d, \quad \theta=\theta^* T^\sharp, \\ \Phi=\Phi^*\Phi^\sharp, \quad T^\sharp=\left(\dfrac{\nu^3\delta T}{Agkd^3}\right)^{1/2}, \\ \Phi^\sharp=\left(\dfrac{\nu^3\delta C}{A_1 gk_1 d^3}\right)^{1/2}, \end{cases} \tag{2.5}$$

Eq. (2.1) (omitting the asterisks) reduces to

$$\begin{cases} P_r^{-1}\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=-\nabla\pi+\Delta\mathbf{u}+\mathcal{T}\mathbf{u}\times\mathbf{k}+(R\theta-R_1\Phi)\mathbf{k}, \\ \nabla\cdot\mathbf{u}=0, \\ \theta_t+P_r\mathbf{u}\cdot\nabla\theta=Rw+\Delta\theta, \\ P_1(\Phi_t+P_r\mathbf{u}\cdot\nabla\Phi)=R_1 w+\Delta\Phi, \end{cases} \tag{2.6}$$

where

$$\begin{cases} R^2=\dfrac{Agd^3\delta T}{\nu k} & \text{thermal Rayleigh number}, \\ R_1^2=\dfrac{A_1 gd^3\delta CP_1}{\nu k} & \text{salt Rayleigh number}, \\ \mathcal{T}^2=\dfrac{4\omega^2 d^4}{\nu^2} & \text{Taylor number}, \\ P_r=\dfrac{\nu}{k} & \text{fluid Prandtl number}, \\ P_1=\dfrac{k}{k_1} & \text{salt Prandtl number}. \end{cases}$$

To (2.6) the boundary conditions (free-free case) are appended

$$\frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}=w=\theta=\Phi=0 \quad \text{on } z=0,1, \tag{2.7}$$

with $\mathbf{u}=(u,v,w)$. We assume (as usually done, in stability problems in layers) that:

(i) the perturbations $(\nabla\pi,u,v,w,\theta,\Phi)$ are periodic in the x and y directions, respectively of periods $2\pi/a_x$, $2\pi/a_y$;
(ii) $\Omega=[0,2\pi/a_x]\times[0,2\pi/a_y]\times[0,1]$ is the periodicity cell;
(iii) $u,v,w,\ \theta,\ \Phi$ are such that together with all their first derivatives and second spatial derivatives are square integrable in Ω, $\forall t\in\mathbb{R}^+$ and can be expanded in a Fourier series uniformly convergent in Ω.

Let us denote by $\mathcal{A}(\Omega)$ the set of functions Ψ such that:

(1) $\Psi:(\mathbf{x},t)\in\Omega\times\mathbb{R}^+\to\Psi(\mathbf{x},t)\in\mathbb{R}$, $\Psi\in W^{2,2}(\Omega)$, $\forall t\in\mathbb{R}^+$, Ψ is periodic in the x and y directions of period $\frac{2\pi}{a_x}$, $\frac{2\pi}{a_y}$ respectively and $(\Psi)_{z=0}=(\Psi)_{z=1}=0$;
(2) Ψ, together with all the first derivatives and second spatial derivatives, can be expanded in a Fourier series absolutely uniformly convergent in Ω, $\forall t\in\mathbb{R}^+$

and let us denote by $\mathcal{B}(\Omega)$ the set of the functions φ such that

(1)′ $\varphi:(\mathbf{x},t)\in\Omega\times\mathbb{R}^+\to\varphi(\mathbf{x},t)\in\mathbb{R}$, $\varphi\in W^{2,2}(\Omega)$, $\forall t\in\mathbb{R}^+$, φ is periodic in the x and y directions of period $\frac{2\pi}{a_x}$, $\frac{2\pi}{a_y}$ respectively and $\left[\frac{\partial\varphi}{\partial z}\right]_{z=0}=\left[\frac{\partial\varphi}{\partial z}\right]_{z=1}=0$;
(2)′ φ, together with all the first derivatives and second spatial derivatives, can be expanded in a Fourier series absolutely uniformly convergent in Ω, $\forall t\in\mathbb{R}^+$.

Since the sequence $\{\sin n\pi z\}_{n\in\mathbb{N}}$ is a complete orthogonal system for $L^2(0,1)$ under the boundary conditions $[\Psi]_{z=0}=[\Psi]_{z=1}=0$, by virtue of periodicity, it turns out that $\forall\Psi\in\mathcal{A}(\Omega)$, there exists a sequence $\{\tilde{\Psi}_n(x,y,t)\}_{n\in\mathbb{N}}$ ($\tilde{\Psi}_n$ being of "plane form") such that

$$\begin{cases} \Psi=\displaystyle\sum_{n=1}^{\infty}\Psi_n=\sum_{n=1}^{\infty}\tilde{\Psi}_n\sin n\pi z, \quad \frac{\partial\Psi}{\partial t}=\sum_{n=1}^{\infty}\frac{\partial\tilde{\Psi}_n}{\partial t}\sin n\pi z, \\ \Delta_1\Psi=-a^2\Psi, \quad \Delta\Psi=-\displaystyle\sum_{n=1}^{\infty}\xi_n\tilde{\Psi}_n\sin n\pi z, \end{cases} \tag{2.8}$$

with $\Delta_1=\partial^2/\partial x^2+\partial^2/\partial y^2$ and

$$\xi_n=a^2+n^2\pi^2, \quad a^2=a_x^2+a_y^2, \tag{2.9}$$

the series appearing in (2.8) being absolutely uniformly convergent in Ω.

Analogously, since the sequence $\{\cos n\pi z\}_{n\in\mathbb{N}}$ is a complete orthogonal system for $L^2(0,1)$ under the boundary conditions $\left[\frac{\partial\varphi}{\partial z}\right]_{z=0}=\left[\frac{\partial\varphi}{\partial z}\right]_{z=1}=0$, by virtue of periodicity, it turns out that $\forall\varphi\in\mathcal{B}(\Omega)$, there exists a sequence $\{\tilde{\varphi}_n(x,y,t)\}_{n\in\mathbb{N}}$ ($\tilde{\varphi}_n$ being of "plane form") such that

$$\begin{cases} \varphi=\displaystyle\sum_{n=1}^{\infty}\varphi_n=\sum_{n=1}^{\infty}\tilde{\varphi}_n\cos n\pi z, \quad \frac{\partial\varphi}{\partial t}=\sum_{n=1}^{\infty}\frac{\partial\tilde{\varphi}_n}{\partial t}\cos n\pi z, \\ \Delta_1\varphi=-a^2\varphi, \quad \Delta\varphi=-\displaystyle\sum_{n=1}^{\infty}\xi_n\tilde{\varphi}_n\cos n\pi z. \end{cases} \tag{2.10}$$

Setting

$$\zeta=(\nabla\times\mathbf{u})\cdot\mathbf{k}=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}, \tag{2.11}$$

the horizontal components of $\mathbf{u}$ are given by

$$u=\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial x\partial z}+\frac{\partial\zeta}{\partial y}\right), \quad v=\frac{1}{a^2}\left(\frac{\partial^2 w}{\partial y\partial z}-\frac{\partial\zeta}{\partial x}\right) \tag{2.12}$$

and – in view of $\mathbf{u}=\sum_{n=1}^{\infty}\mathbf{u}_n$, $\zeta_n=\frac{\partial v_n}{\partial x}-\frac{\partial u_n}{\partial y}$ – it follows that

$$\begin{cases} u_n=\dfrac{1}{a^2}\left(\dfrac{\partial^2 w_n}{\partial x\partial z}+\dfrac{\partial\zeta_n}{\partial y}\right), \quad v_n=\dfrac{1}{a^2}\left(\dfrac{\partial^2 w_n}{\partial y\partial z}-\dfrac{\partial\zeta_n}{\partial x}\right), \\ \nabla\cdot\mathbf{u}_n=\left(\dfrac{1}{a^2}\Delta_1 w_n+w_n\right)_z=0 \end{cases} \tag{2.13}$$

Remark 2.1. We remark that, in view of $(2.10)_1$, one has

$$\int_\Omega u\,d\Omega\equiv\int_\Omega v\,d\Omega\equiv 0. \tag{2.14}$$

Remark 2.2. Let us denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the $L^2(\Omega)$–norm and the scalar product, respectively. Multiplying $(2.6)_1$ for $\mathbf{u}$ and integrating over Ω, one has

$$\frac{P_r^{-1}}{2}\frac{d}{dt}\|\mathbf{u}\|^2=-\|\nabla\mathbf{u}\|^2+\langle R\theta,w\rangle-\langle R_1\Phi,w\rangle. \tag{2.15}$$