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# A unified compatibility method for exact solutions of non-linear flow models of Newtonian and non-Newtonian fluids

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ABSTRACT

Criteria are established for higher order ordinary differential equations to be compatible with lower order ordinary differential equations. Necessary and sufficient compatibility conditions are derived which can be used to construct exact solutions of higher order ordinary differential equations subject to lower order equations. We provide the connection to generalized groups through conditional symmetries. Using this approach of compatibility and generalized groups, new exact solutions of non-linear flow problems arising in the study of Newtonian and non-Newtonian fluids are derived. The ansatz approach for obtaining exact solutions for non-linear flow models of Newtonian and non-Newtonian fluids is unified with the application of the compatibility and generalized group criteria.


## 1. Introduction

Considerable attention has been given in recent times to finding the exact solutions of differential equations describing the behavior of non-Newtonian fluids. Due to the increasing importance of non-Newtonian fluids in innovative technology and modern industries, the motivation of investigators to study problems dealing with the flow of non-Newtonian fluids has increased enormously. Modelling non-Newtonian flows is important for understanding and predicting the behavior of processes and thus for designing optimal flow configurations and for selecting conditions.

Fluids are classified into two broad categories: Newtonian fluids and non-Newtonian fluids. Fluids which satisfy a linear relation between the stress and rate-of-strain are classified as Newtonian fluids. However, most fluids in industry do not adhere to the commonly accepted assumption of a linear relationship between the stress and the rate-of-strain and thus are characterized as non-Newtonian fluids. The flow behavior of non-Newtonian fluids is quite different from that of Newtonian fluids. Therefore, in practical applications one cannot replace the behavior of non-Newtonian fluids with that of a Newtonian fluid. It is important to understand the physical behavior of non-Newtonian fluids in order to improve their utilization in various manufacturing processes.

The most challenging task that we need to address when dealing with flow problems of Newtonian and non-Newtonian fluids is that the governing equations of these models are of a high order and complicated in nature. Such fluids are modelled by constitutive equations which vary greatly in complexity. Thus, the resulting non-linear equations are not easy to solve exactly. These exact solutions, if available, facilitate the verification of numerical codes and are also helpful in a stability analysis. Consequently, exact (closed-form) solutions of flow models of Newtonian and non-Newtonian fluids are important. Several methods have been developed in recent years to obtain the solutions of these fluid models. Some of the techniques are the variational iteration method, Adomian decomposition method, homotopy analysis method, homotopy perturbation method, simplest equation method, semi-inverse variational method and the exponential function method. However, all these methods have certain limitations and in general fail to derive exact closed-form solutions for non-linear models of Newtonian and non-Newtonian fluids.

A significant extension of classical Lie symmetry groups for partial differential equations (PDEs) is that of non-classical symmetry or generalized group analysis [1,2]. For PDEs, Olver and Rosenau [3] have demonstrated by means of examples that the construction of various special solutions to PDEs can effectively be deduced by a single method consisting of adding one or more side conditions to the PDE under investigation. The side conditions are always chosen to simplify the study of the PDE in question. Pucci and Saccomandi [4] state that "the classical method employed in fluid dynamics of finding exact solutions is an application of this idea". The side

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conditions which express the invariance of the PDE solutions with respect to a continuous group of point transformations are studied in [4] where an algorithm is defined to characterize weak symmetry groups of PDEs. Here we consider a unifying approach to the problem of side conditions for ODEs that arise in fluid mechanics. The conditional symmetries of ODEs from the algorithmic viewpoint have been discussed in detail by Fatima and Mahomed [5] as well as Mahomed and Qadir [6]. This allows one to construct particular compatible side conditions to the given ODE in order to deduce exact solutions if there is a sufficient supply of conditional symmetries. We utilize the compatibility and conditional symmetry approach for the exact solvability of ODEs arising from fluid mechanics and provide a unified treatment.

In recent years, the ansatz method has been used to construct exact solutions of non-linear differential equations arising in the study of Newtonian and non-Newtonian fluids. In the ansatz method different forms of the solution are assumed and different techniques are used to develop analytical results. In this paper, we present a unified treatment to classify exact solutions of models of Newtonian and non-Newtonian fluids which are solved using different techniques [7–41]. We develop a general compatibility and conditional symmetry approach and then apply it to several existing studies in a unified manner. We construct some new exact solutions and also reconstruct some existing exact solutions of non-linear problems for Newtonian and non-Newtonian fluids using the concept of compatibility and generalized group. Some of these models were previously solved by the ansatz approach but not in a systematic way.

In this paper, we first give a rigorous definition of compatibility and then developed a general compatibility test for a higher order ordinary differential equation (up to fifth-order) to be compatible with a first order ordinary differential equation. We then connect this to generalized groups, viz. conditional symmetry groups. This can be extended in a natural way to higher order with existing computer codes. Several examples [7–41] from the literature are presented to which the compatibility and group approach is applied and new as well as existing exact solutions are deduced from a single viewpoint in a unified manner.

## 2. Compatibility approach

It is often difficult to obtain exact solutions of a higher order non-linear differential equation. For this reason, many researchers have assumed a form of the exact solution by trial and error. We provide a general compatibility and generalized group criteria for many of these higher order ordinary differential equations arising in the study of Newtonian and non-Newtonian fluid models. This compatibility and group approach leads to some new exact solutions of these models and is also helpful in reproducing existing solutions. Thus the approach here unifies the explicit solution construction for all ansatz approaches considered.

Firstly, we present a precise definition of compatibility.

**Definition 1.** Consider the $n$th-order ordinary differential equation:

$$y^{(n)}=P(x,y,y^{(1)},y^{(2)},\ldots,y^{(n-1)}),\quad n\geq 2, \tag{1}$$

where $x$ is the independent variable, $y$ the dependent variable and $y^{(1)},y^{(2)},\ldots,y^{(n)}$ denote the first, second,…,$n$th derivative of $y$ with respect to $x$. If every solution of the $m$th-order ordinary differential equation:

$$y^{(m)}=Q(x,y,y^{(1)},y^{(2)},\ldots,y^{(m-1)}),\quad m<n, \tag{2}$$

is also a solution of the $n$th-order ordinary differential Eq. (1), then the $m$th-order ordinary differential Eq. (2) is said to be compatible with the $n$th-order ordinary differential Eq. (1).

We will consider the compatibility of a $n$th-order ($n\geq 2$) ordinary differential equation with a first order ordinary differential equation so that $m=1$.

### 2.1. Compatibility criterion for a fifth order ordinary differential equation

Here we develop a compatibility criterion or compatibility test for a fifth order ordinary differential equation to be compatible with a first order ordinary differential equation.

Let us consider a fifth-order ordinary differential equation in one independent variable $x$ and one dependent variable $y$:

$$F(x,y,y^{(1)},y^{(2)},\ldots,y^{(5)})=0, \tag{3}$$

and a first order ordinary differential equation:

$$E(x,y,y^{(1)})=0, \tag{4}$$

such that

$$J=\frac{\partial[E,F]}{\partial[y^{(1)},y^{(2)},\ldots,y^{(5)}]}\neq 0. \tag{5}$$

Then, we can solve for the highest derivatives as

$$y^{(5)}=f(x,y,y^{(1)},y^{(2)},y^{(3)},y^{(4)}), \tag{6}$$

and

$$y^{(1)}=e(x,y), \tag{7}$$

where $f$ and $e$ are smooth and continuously differentiable functions of $x$, $y$ and, in the case of $f$, the derivatives of $y$.

Now Eq. (6) depends on $y^{(1)},\ldots,y^{(5)}$ which are obtained by differentiating Eq. (7). This gives

$$y^{(2)}=e_x+ee_y, \tag{8}$$

$$y^{(3)}=e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2, \tag{9}$$

$$\begin{aligned}y^{(4)}&=e_{xxx}+3ee_{xxy}+3e^2e_{xyy}+e^3e_{yyy}+3ee_{yy}(e_x+ee_y)\\&\quad+3e_{xy}(e_x+ee_y)+e_y(e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2)\end{aligned} \tag{10}$$

$$\begin{aligned}y^{(5)}&=e_{xxxx}+4ee_{xxxy}+6e^2e_{xxyy}+4e^3e_{xyyy}+e^4e_{yyyy}+6e^2e_{yyy}\{e_x+ee_y\}\\&\quad+12ee_{xyy}\{e_x+ee_y\}+4ee_{yy}\{e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2\}\\&\quad+5e_{xxy}\{e_x+ee_y\}+4e_{xy}\{e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2\}\\&\quad+e_y[e_{xxx}+3ee_{xxy}+3e^2e_{xyy}+e^3e_{yyy}+3ee_{yy}\{e_x+ee_y\}\\&\quad+3e_{xy}\{e_x+ee_y\}+e_y\{e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2\}]+3e^2e_{yy}.\end{aligned} \tag{11}$$

By equating the right hand side of Eq. (6) with Eq. (11), we obtain

$$\begin{aligned}&f[x,y,e,(e_x+ee_y),\left(e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2\right),\\&\quad(e_{xxx}+3ee_{xxy}+3e^2e_{xyy}+e^3e_{yyy}+3ee_{yy}(e_x+ee_y)\\&\quad+3e_{xy}(e_x+ee_y)+e_y(e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2))]\\&=e_{xxxx}+4ee_{xxxy}+6e^2e_{xxyy}+4e^3e_{xyyy}+e^4e_{yyyy}+6e^2e_{yyy}\{e_x+ee_y\}\\&\quad+12ee_{xyy}\{e_x+ee_y\}+4ee_{yy}\{e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2\}\\&\quad+5e_{xxy}\{e_x+ee_y\}+4e_{xy}\{e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2\}\\&\quad+e_y[e_{xxx}+3ee_{xxy}+3e^2e_{xyy}+e^3e_{yyy}+3ee_{yy}\{e_x+ee_y\}\\&\quad+3e_{xy}\{e_x+ee_y\}+e_y\{e_{xx}+2ee_{xy}+e^2e_{yy}+e_xe_y+ee_y^2\}]\\&\quad+3e^2e_{yy},\end{aligned} \tag{12}$$

which gives the general compatibility criterion or compatibility test for a fifth order ordinary differential equation to be compatible with a first order ordinary differential equation.