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Extended models of non-linear waves in liquid with gas bubbles

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ABSTRACT

In this work we generalize the models for non-linear waves in a gas–liquid mixture taking into account an interphase heat transfer, a surface tension and a weak liquid compressibility simultaneously at the derivation of the equations for non-linear waves. We also take into consideration high order terms with respect to the small parameter. Two new non-linear differential equations are derived for long weakly non-linear waves in a liquid with gas bubbles by the reductive perturbation method considering both high order terms with respect to the small parameter and the above-mentioned physical properties. One of these equations is the perturbation of the Burgers equation and corresponds to main influence of dissipation on non-linear waves propagation. The other equation is the perturbation of the Burgers–Korteweg–de Vries equation and corresponds to main influence of dispersion on non-linear waves propagation.

1. Introduction

A liquid with gas bubbles has many applications in nature, industry and medicine [1,2]. Non-linear wave processes in a gas–liquid mixture were studied for the first time in works [3–5]. The Burgers, the Korteweg–de Vries and the Burgers–Korteweg–de-Vries equations were obtained in [3–5] for the description of long weakly non-linear waves. The fourth-order non-linear evolution equation for non-linear waves in a gas–liquid mixture was obtained in [6,7] taking into account an interphase heat transfer. Non-linear waves in a liquid with gas bubbles in the three-dimensional case were considered in [8]. Linear waves in a gas–liquid mixture under the van Wijngaarden's theory were studied in [9,10]. In [11] propagation of linear waves in a liquid containing gas bubbles at finite volume fraction was considered.

In the previous studies of non-linear waves in a liquid containing gas bubbles only the first-order terms with respect to the small parameter were taken into account. On the other hand we know that using high order terms with respect to the small parameter at the derivation of non-linear evolution equations allows us to obtain a more exact description of non-linear waves [12–23]. Also taking into account high order correction in equations for non-linear waves one can reveal important physical phenomena, such as interaction between dissipative and dispersive processes in a gas–liquid mixture and its influence on waves propagation, new mechanisms of waves dispersion and dissipation. Thus, it is important to study non-linear waves in a liquid with gas bubbles taking into account second-order terms in the asymptotic expansion.

We investigate non-linear waves in a liquid with gas bubbles taking into consideration not only high order terms with respect to the small parameter but the surface tension, liquid viscosity, interphase heat transfer and weak liquid compressibility as well. To the best of our knowledge the influence of these physical properties on non-linear waves propagation simultaneously was not considered previously.

The aim of our work is to study long weakly non-linear waves in a liquid with gas bubbles taking into account both high order terms in the asymptotic expansion and the above-mentioned physical properties in the model for non-linear waves. We use the reductive perturbation method for the derivation of differential equations for non-linear waves.

We apply the concept of the asymptotic equivalence, asymptotic integrability and near-identity transformations [12,13,15,17,18] for studying non-linear equations for long waves in a gas–liquid mixture. Asymptotically equivalent equations obtained in this work are connected to each other by a continuous group of non-local transformations [17,18]. These transformations are near-identity transformations. Thus, we introduce families of asymptotically equivalent equations for long weakly non-linear waves in a liquid containing gas bubbles at quadratic order. As far as all these equations are equivalent we can use a more convenient and simple equation within this family. This equation is a normal form equation. Such approach for the investigation of non-linear evolution equation was proposed in works [12,13]. Near-identity transformations are often named Kodama's transformations.

We derive two new non-linear differential equations for long weakly non-linear waves in a liquid with gas bubbles by the reductive

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perturbation method. In the case of dissipation main influence non-linear waves are governed by the perturbation of the Burgers equation. The perturbation of the Burgers–Korteweg–de Vries equation corresponds to the main influence of dispersion on non-linear waves propagation.

We analyze dispersion relations for both equations. Near-identity transformations are used to obtain normal forms for the above-mentioned equations. We show that a normal form for the equation in the dissipative case can be linearized under a certain condition on physical parameters. It is worth noting that this condition is realizable for physically meaningful values of parameters. Analytical solution of the general dissipative equation in the form of a weak shock wave is obtained and analyzed.

Two cases of a normal form equation are analyzed provided that dispersion has the main influence. The first one is the case of negligible dissipation (purely dispersive case) where non-linear waves are governed by the generalized Korteweg–de Vries equation [24]. We show that the generalized Korteweg–de Vries equation for non-linear waves in a liquid with gas bubbles is asymptotically equivalent to one of the integrable fifth-order evolution equations that are the Lax, the Sawada–Kotera and the Kaup–Kupershmidt equations. The general form of the dispersive non-linear evolution equation seems to be non-integrable. However, this equation admits analytical solitary wave solutions.

The rest of this work is organized as follows. In Section 2 we give the basic system of equations for non-linear waves in a liquid with gas bubbles. We discuss the dispersion relation for linear waves as well. The main non-linear differential equation for long weakly non-linear waves is obtained by the reductive perturbation method. The non-linear waves with the main influence of dissipation are studied in Section 3. Section 4 is devoted to the investigation of non-linear waves in the case of dispersion main influence. In Section 5 we briefly discuss our results.

2. Main differential equation for long weakly non-linear waves in a liquid with gas bubbles

For studying non-linear waves in a liquid with gas bubbles we use the homogeneous model [1,2]. We consider a bubble–liquid mixture as a homogeneous media with an average pressure, an average density and an average velocity. We do not take into account interaction, formation, destruction and coalescence of bubbles. Thus, the amount of gas bubbles in the mass unit is the constant N. We assume that all gas bubbles are spherical. The nearly isothermal approximation [25] is used for the modeling of heat transfer between a gas in bubbles and a liquid. In this approximation it is supposed that the temperature of the liquid is not changed and is equal to the temperature of the mixture in the unperturbed state (T_0) [25]. We consider influence of the liquid viscosity only at the interphase boundary. Also we take into consideration the weak compressibility of the liquid using the Keller–Miksis equation for the description of bubbles dynamics [26,27]. Also we consider the one-dimensional case. In these assumptions we can use the following system of equations for the description of non-linear waves in the liquid with gas bubbles [1,2]:

$$\rho_\tau+(\rho\tilde{u})_\xi=0, \tag{1a}$$

$$\rho(\tilde{u}_\tau+\tilde{u}\tilde{u}_\xi)+P_\xi=0, \tag{1b}$$

$$\rho_l\left(RR_{\tau\tau}+\frac{3}{2}R_\tau^2+\frac{4\nu_l}{3R}R_\tau\right)-\frac{\rho_l}{c_l}(R^2R_{\tau\tau\tau}+6RR_\tau R_{\tau\tau}+R_\tau^3)=P_g-P-\frac{2\sigma}{R}, \tag{1c}$$

$$P_g=P_{g,0}\left(\frac{R_0}{R}\right)^3\left\{1-\frac{(\gamma_g-1)R_0^3}{5\gamma_g\chi_g}\frac{R_\tau}{R^2}-\frac{(\gamma_g-1)^2R_0^6}{525\gamma_g^2\chi_g^2}\times\left[(2+15K_0')\frac{R_\tau^2}{R^4}+\frac{12\gamma_g-7}{3(\gamma_g-1)R}\left(\frac{R_{\tau\tau}}{R^2}-\frac{2R_\tau^2}{R^3}\right)\right]\right\}, \tag{1d}$$

$$\rho=(1-\phi)\rho_l+\phi\,\rho_g, \tag{1e}$$

$$\phi=V\rho,\quad V=\frac{4}{3}\pi R^3N. \tag{1f}$$

We use the following notations in system (1): ξ is the cartesian coordinate, τ is the time, $\rho(\xi,\tau)$ is the density of the bubble–liquid mixture, $P(\xi,\tau)$ is the pressure of the mixture, $\tilde{u}(\xi,\tau)$ is the velocity of the mixture, $R=R(\xi,\tau)$ is the bubbles radius, $\rho_l,\rho_g(\xi,\tau)$ are the densities of the liquid and the gas respectively, $P_g(\xi,\tau)$ is the pressure of the gas in bubbles, $P_{g,0}$ and R_0 are the pressure and the radius of bubbles in the unperturbed state respectively, ϕ is the volume gas content, V is the specific volume of the gas in the mixture, σ is the surface tension, ν_l is the kinematic liquid viscosity, γ_g is the ratio of the specific heats for the gas, χ_g is the thermal diffusivity of the gas, K_g is the thermal conductivity of the gas, $K_0'=dK/dT$ at $T=T_0$, where T_0 is the temperature of the mixture in the unperturbed state.

The first two equations from system (1) are the continuity equation and the Euler equation for the mixture. Let us note that at the derivation of Eq. (1c) for bubbles' dynamics the liquid viscosity at the inter-phase boundary, the slight liquid compressibility and surface tension were taken into consideration [26,27]. Eq. (1d) was obtained in [25] under assumption that the gas temperature in the bubble deviates little from the temperature in the unperturbed state. Eqs. (1e) and (1f) are definitions of the gas–liquid mixture density, the volume gas content and the specific volume of the gas in the mixture correspondingly.

Let us note that the approach based on the theory of thermo-microstretch fluid [28] can be used for the description of a gas–liquid continuum. For example, acceleration waves in a thermo-microstretch fluid were studied in [28].

We suppose that the pressure and density of the mixture in the unperturbed state are constants and all bubbles have the same radius and are uniformly distributed in the liquid.

Assuming that the volume gas content is small $\phi\ll 1$ from (1e) to (1f) we obtain

$$\rho=\frac{\rho_l}{1+\rho_l V},\quad V=\frac{4}{3}\pi R^3N. \tag{2}$$

This equation connects the density of the bubble–liquid mixture with the bubbles radius.

We use the following initial conditions:

$$t=0:P=P_g=P_0,\quad P_0=\text{const}.$$

Let us suppose that deviation of the mixture density is small:

$$\rho(\xi,\tau)=\rho_0+\delta\tilde{\rho}(\xi,\tau),\quad \rho_0=\text{const},\quad \delta=\frac{\|\rho-\rho_0\|}{\rho_0}\ll 1,$$
$$\rho(\xi,0)=\rho_0, \tag{3}$$

where δ is a small parameter corresponding to small deviations of the mixture density from its equilibrium value.

Using formula (3) from (2) with accuracy up to δ^2 we obtain

$$R=R_0-\chi\delta\tilde{\rho}+\chi_1\delta^2\tilde{\rho}^2,$$
$$R_0^3=\frac{3}{4\pi N}\left(\frac{1}{\rho_0}-\frac{1}{\rho_l}\right),\quad \chi=\frac{R_0}{3\,\rho_0^2\,V_0},\quad \chi_1=\frac{R_0(3\,\rho_0V_0-1)}{9\,\rho_0^4\,V_0^2}. \tag{4}$$

Substituting (3) and (4) into Eqs. (1a)–(1d) and using the dimensionless variables

$$\xi=L\,\xi',\quad \tau=\frac{L}{c_0}\tau',\quad \tilde{u}=\delta c_0\,\tilde{u}',\quad \tilde{\rho}=\rho_0\tilde{\rho}',\quad P=\delta P_0\,P'+P_0-\frac{2\sigma}{R_0}, \tag{5}$$