International Journal of Non-Linear Mechanics 62 (2014) 33–40

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Non-linear instability and exact solutions to some delay reaction–diffusion systems

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ARTICLE INFO

Article history:
Received 5 December 2013
Received in revised form
7 February 2014
Accepted 11 February 2014
Available online 18 February 2014

Keywords:
Delay reaction–diffusion systems
Non-linear instability
Global instability
Delay partial differential equations
Ill-posed problems
Exact solutions

ABSTRACT

We deal with coupled delay non-linear reaction–diffusion systems of the form

$$u_t = k_1 u_{xx} + F(u, \overline{u}, w, \overline{w}),$$
$$w_t = k_2 w_{xx} + G(u, \overline{u}, w, \overline{w}),$$

where $u = u(x,t)$, $w = w(x,t)$, $\overline{u} = u(x, t-\tau)$, and $\overline{w} = w(x, t-\tau)$, and τ is the delay time. For a wide class of the kinetic functions F and G, we determine global instability conditions; once these conditions hold, any solution of the system is unstable. The solution instability is proved with an exact approach without making any assumptions or approximations (this approach can be useful for analyzing other non-linear delay models, including biological, biochemical, biophysical, etc.). We discuss some ill-posed Cauchy-type and initial-boundary-value problems connected with the global instability. We present multi-parameter exact solutions involving an arbitrary number of free parameters and give an exact solution that represents a non-linear superposition of a traveling wave and a periodic standing wave. All of the systems considered contain two arbitrary functions of two or three arguments. We also study other non-linear systems of delay PDEs including reaction–diffusion system with two different time-varying delays, multicomponent systems of reaction–diffusion equations and more complex, higher-order non-linear systems with delay. The results may be used in solving certain problems and testing approximate analytical and numerical methods for certain classes of similar and more complex non-linear delay systems.

1. Introduction. Classes of systems considered. Remarks

1.1. *Preliminary remarks. Single non-linear delay reaction–diffusion equations*

Non-linear delay reaction–diffusion equations and coupled systems of such equations arise in biology, biophysics, biochemistry, chemistry, medicine, control, climate model theory, ecology, economics, and many other areas (e.g., see the studies [1–10] and references in them). Similar equations occur in the mathematical theory of artificial neural networks and the results are used for signal and image processing as well as in image recognition problems [11–20].

To begin with, we give brief introductory information and outline some results for the single non-linear delay reaction–diffusion equation

$$u_t = k u_{xx} + F(u, \overline{u}), \quad \overline{u} = u(x, t-\tau). \tag{1}$$

The rate of change of the unknown in biochemical, biological, physico-chemical, ecological, and other systems generally depends not only on the current state of the system at a fixed time but also on its entire previous evolution or on the values of the unknown at certain times in the past. The latter case is modeled by equations of the form (1), where the kinetic function F (rate of chemical or biochemical reaction) depends on both $u = u(x,t)$ and its delayed counterpart $\overline{u} = u(x, t-\tau)$. If $F(u, \overline{u}) = f(\overline{u})$, the delay suggests, from the physical viewpoint, that mass/heat transfer processes in local non-equilibrium media possess inertia: the system does not respond to an action immediately at the time t when the action is applied, which is the case in the classical local-equilibrium case, but a delay time τ later.

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http://dx.doi.org/10.1016/j.ijnonlinmec.2014.02.003
0020-7462

The delay τ in reaction–diffusion equations and other non-linear PDEs may be due to a number of different factors depending on the area of application. For example, in biology and biomechanics, the delay may be associated with the finiteness of the speed of neural response in living tissues. In medicine, in decease development problems, the delay time is determined by the incubation period during which the disease develops (in certain cases, one should also take into account the time after which an infected animal becomes contagious). In population dynamics, the delay is due to a gestation or maturation period. In control theory, the delay usually results from the finiteness of the signal processing speed and rate of technological processes.

Remark 1. A number of exact solutions to the heat equation with a non-linear source, which is a special case of Eq. (1) where there is no delay and $F(u,\overline{u})=f(u)$, can be found, for example, in [21–25]. A comprehensive survey of exact solutions to this equation and more complex non-linear heat equations without delay can be found in the handbook [26].

The presence of delay in Eq. (1) makes it much more difficult to investigate than non-linear PDEs without delay.

In general, Eq. (1) admits obvious traveling-wave solutions, $u=u(\alpha x+\beta t)$. Such solutions are dealt with in many studies (e.g., see the papers [2–7] and references in them). A complete group analysis of the non-linear delay equation (1) was carried out in [27]; four equations of the form (1) were found to admit invariant solutions (two of these equations have degenerate solutions that are linear in x). Recently, the studies [28–31] described a number of simple separable, generalized separable, and functional separable solutions to equations of the form (1) as well as several exact solutions to more complex non-linear reaction–diffusion equations with time-varying delay, $\tau=\tau(t)$ (the majority of the equations considered contain one or two arbitrary functions of a single argument).

Remark 2. The following two modified Fisher equations (modified diffusion logistic equations) with delay are examples of non-linear delay reaction–diffusion equations of the form (1) with quadratic non-linearity:

$$u_t=ku_{xx}+bu-u(cu+a\overline{u}), \tag{2}$$

$$u_t=ku_{xx}+bu-c(u-a\overline{u})^2. \tag{3}$$

At $a=0$ or $\tau=0$, these become the Fisher–KPP equation without delay [32,33]. The delay Fisher equation (2) with $c=0$ was studied in [3,4]. Some exact solutions to more complex equations, including (2) and (3), can be found in [28,30].

1.2. Non-linear delay reaction–diffusion systems considered

The present paper deals with non-linear reaction–diffusion systems of two coupled delay equations of the form

$$u_t=k_1u_{xx}+F(u,\overline{u},w,\overline{w}), \tag{4}$$

$$w_t=k_2w_{xx}+G(u,\overline{u},w,\overline{w}), \tag{5}$$

where $u=u(x,t)$, $w=w(x,t)$, $\overline{u}=u(x,t-\tau)$, and $\overline{w}=w(x,t-\tau)$, and τ is the delay time ($k_1>0$, $k_2>0$, and $\tau>0$). Besides (4)–(5), we also consider more complex non-linear systems of coupled delay PDEs (see Sections 4.1 and 4.2).

Remark 3. A considerable number of exact solutions to non-linear systems of the form (4)–(5) without delay ($\tau=0$) and more complex non-linear reaction–diffusion systems without delay can be found, for example, in [26,34–39].

Remark 4. A system of two coupled delay reaction–diffusion equations with quadratic non-linearity is used to model Belousov–Zhabotinskii reaction [3].

In general, system (4)–(5) admits traveling-wave solutions $u=u(z)$, $w=w(z)$ with $z=\alpha x+\beta t$, where α and β are arbitrary constants. The issues of stability (usually linear) of time-invariant, traveling wave, and some other solutions to various delay reaction–diffusion equations and systems of equations are addressed in numerous studies, for example, [2,7–20].

1.3. The concept of 'exact solution' for non-linear delay PDEs. Remarks

In what follows, the term 'exact solution' with regard to non-linear delay partial differential equations and systems of delay PDEs is used in the following cases:

(i) the solution is expressible in terms of elementary functions or in closed form with definite or indefinite integrals;
(ii) the solution is expressible in terms of solutions to ordinary differential or delay ordinary differential equations (or systems of ODEs or delay ODEs);
(iii) the solution is expressible in terms of solutions to linear partial differential equations (or systems of linear PDEs).

Combination of cases (i)–(iii) is also allowed.

This definition generalizes the notion of an exact solution used in [26] with regard to non-linear partial differential equations.

Remark 5. For solution methods and various applications of linear and non-linear delay ordinary differential equations, which are much simpler than non-linear delay partial differential equations, see, for example, [40–45].

It is important to note that currently there are no studies published on exact solutions (other than traveling-wave solutions) to non-linear reaction–diffusion systems of coupled delay equations.

Remark 6. Exact solutions to some non-linear delay PDEs (and systems of non-linear delay PDEs) other than (1) and (4)–(5) can be found, for example, in [28,46–48].

Remark 7. For numerical solution methods for non-linear reaction–diffusion systems of coupled delay equations and other non-linear systems of delay PDEs as well as related difficulties, see [49–52]. The exact solutions presented in this paper can be used as test problems for independent verification of numerical methods for non-linear systems of delay PDEs.

1.4. Principal new results presented in the paper

The paper will present the following principal results:

- Exact solutions to coupled delay non-linear reaction–diffusion systems other than traveling-wave solutions have been obtained for the first time.
- Global instability conditions have been obtained for solutions to a wide class of non-linear reaction–diffusion systems of delay equations involving two arbitrary functions of three arguments.
- It has been shown that if the global instability conditions hold for a reaction–diffusion system, then Cauchy-type and some initial-boundary-value problems for this system are ill-posed with respect to the initial data.
- Exact solutions to a coupled non-linear reaction–diffusion system with two different time-dependent delays, $\tau_1=\tau_1(t)$ and $\tau_2=\tau_2(t)$, have been described for the first time.