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The interplay of group and dynamical systems analysis: The case of spherically symmetric charged fluids in general relativity

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ABSTRACT

We investigate the relationship between the Dynamical Systems analysis and the Lie Symmetry analysis of ordinary differential equations. We undertake this investigation by looking at a relativistic model of self-gravitating charged fluids. Specifically we look at the impact of specific parameters obtained from Lie Symmetries analysis on the qualitative behaviour of the model. Steady states, stability and possible bifurcations are explored. We show that, in some cases, the Lie analysis can help to simply the dynamical systems analysis.

1. Introduction

Most of the real world problems (in biology, finance, economics, industry, etc.), and many fundamental laws of physics and chemistry are formulated in the form of differential (or difference) equations. Various methods for solving or analysing such equations have been developed. In the late 19th century, Marius Sophus Lie unified many of these methods by introducing the notion of (what has become known as) Lie groups [23]. The Lie theory of differential equations has been phenomenally successful in determining solutions to differential equations [3,21]. It is a useful tool that can be applied to either find solutions explicitly or it can be used to classify equations via equivalence transformations. A major hurdle has been the oftentimes tedious calculations involved in finding the symmetries. However, with the advent of very capable computer packages [11,6], this disadvantage has been overcome. Symmetries can be easily calculated for a variety of equations and then used to obtain solutions, if possible.

Another very useful approach to differential equations is that of dynamical systems analysis [26] in which the long-term behaviour of a system is investigated by focusing on linearization around equilibrium points. This approach has its genesis in Newtonian mechanics, and emphasizes on qualitative rather than quantitative questions [2,17,22]. For example, it was eventually realized that equations describing the motion of the three-body problem (sun, earth and moon) were difficult to solve analytically [26]. Instead of focusing only of the exact positions of the planets at all times, people looked at their stability. Thus, the qualitative analysis is helpful, particularly when the exact solution of an equation cannot be found (and also can give more useful information even when exact solutions exist).

Though these two approaches look as they belong to different areas of mathematics, they have some structures in common. For instance, in a natural way, the equivariant bifurcation theory can be viewed as an application of Lie groups in symmetric systems [30]. Therefore, it is natural to consider the use of both approaches when analysing any differential equation of interest. In what follows, we present the results of applying both methods to an Emden–Fowler equation of index three. Such equations have a lengthy history [29,19,25,10,14] but, we believe, have not been approached in this two-fold manner. We will show how each method can be used to obtain interesting information about the behaviour of the solutions.

2. The model

The metric governing the shear-free motion of a spherically symmetric perfect fluid can be written as

$$ds^2 = -e^{2\nu(t,r)}\,dt^2 + e^{2\lambda(t,r)}[dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)] \quad (1)$$

where ν and λ are the gravitational potentials. If we impose the presence of an electromagnetic field, the usual Einstein field equations must be modified and supplemented with Maxwell's equations. The resulting Einstein–Maxwell system is then given by

$$\rho = 3\frac{\lambda_t^2}{e^{2\nu}} - \frac{1}{e^{2\lambda}}\left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right) - \frac{E^2}{r^4 e^{4\lambda}} \quad (2)$$

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$$p=\frac{1}{e^{2\nu}}(-3\lambda_t^2-2\lambda_{tt}+2\nu_t\lambda_t)+\frac{1}{e^{2\lambda}}\left(\lambda_r^2+2\nu_r\lambda_r+\frac{2\nu_r}{r}+\frac{2\lambda_r}{r}\right)+\frac{E^2}{r^4e^{4\lambda}} \quad (3)$$

$$p=\frac{1}{e^{2\nu}}(-3\lambda_t^2-2\lambda_{tt}+2\nu_t\lambda_t)+\frac{1}{e^{2\lambda}}\left(\nu_{rr}+\nu_r^2+\frac{\nu_r}{r}+\frac{\lambda_r}{r}+\lambda_{rr}\right)-\frac{E^2}{r^4e^{4\lambda}} \quad (4)$$

$$0=\nu_r\lambda_t-\lambda_{tr} \quad (5)$$

$$E=r^2e^{\lambda-\nu}\Phi_r \quad (6)$$

$$E_r=\sigma r^2e^{3\lambda} \quad (7)$$

Here, ρ is the energy density, p is the isotropic pressure, σ is the proper charge density of the fluid and we interpret E as the total charge contained within the sphere of radius r centred around the origin of the coordinate system. The electromagnetic field is present via Φ_r.

Integrating (5) and combining (3) and (4) and integrating allow us to reduce the system to

$$\rho=3e^{2h}-e^{-2\lambda}\left(2\lambda_{rr}+\lambda_r^2+\frac{4\lambda_r}{r}\right)-\frac{E^2}{r^4e^{4\lambda}} \quad (8)$$

$$p=\frac{1}{\lambda_te^{3\lambda}}\left[e^{\lambda}\left(\lambda_r^2+\frac{2\lambda_r}{r}\right)-e^{3\lambda+2h}-\frac{E^2}{r^4e^{\lambda}}\right]_t \quad (9)$$

$$e^{\nu}=\lambda_te^{-h} \quad (10)$$

$$e^{\lambda}\left(\lambda_{rr}-\lambda_r^2-\frac{\lambda_r}{r}\right)=-\tilde{F}-\frac{2E^2}{r^4e^{\lambda}} \quad (11)$$

$$E=r^2e^{\lambda-\nu}\Phi_r \quad (12)$$

$$E_r=\sigma r^2e^{3\lambda}, \quad (13)$$

where $h=h(t)$ and $\tilde{F}=\tilde{F}(r)$ are arbitrary functions of integration. This means that we only need to solve (11) in order to obtain all the other unknown functions.

Using the transformation [7,15]

$$x=r^2 \quad (14)$$

$$y=e^{-\lambda} \quad (15)$$

$$f(x)=\frac{\tilde{F}}{4r^2} \quad (16)$$

$$g(x)=\frac{E^2}{2r^6} \quad (17)$$

we can rewrite (11) as

$$y''=f(x)y^2+g(x)y^3. \quad (18)$$

Remark. (1) In this approach, our interest is only in point symmetries. We acknowledge that a variety of other symmetries exist, including potential [4], Lie–Bäcklund/generalized [1,21], Lambda [20,24,8] and non-local [9]. Expanding our study to those symmetries may indeed yield other useful results but they are outside the scope of our work.

(2) In general, (18) does not admit any Lie point symmetries. As a result, it cannot be linearized under a point transformation. In what follows, we investigate under what conditions the equation *does* admit point symmetries. We first look at conditions under which it admits a single Lie point symmetry (and so is part of the equivalence class of autonomous second order differential equations). Thereafter, we investigate additional conditions under which the equation admits a second Lie point symmetry. This is important as the possession of two Lie point symmetries guarantees that the equation can be reduced to quadratures.

Eq. (18) is the fundamental non-linear ordinary differential equation which determines the behaviour of self-gravitating charged fluids in general relativity [15]. Once y is determined, Kweyama et al. [15] found that

$$G=a\frac{\partial}{\partial x}+(by+c)\frac{\partial}{\partial y} \quad (19)$$

is a point symmetry of (18), provided the functions $a(x)$, $b(x)$, and $c(x)$ satisfied the following system of ordinary differential equations:

$$a''=2b' \quad (20)$$

$$b''=2fc \quad (21)$$

$$c''=0 \quad (22)$$

$$af'+(2a'+b)f=-3cg \quad (23)$$

$$ag'+(2a'+2b)g=0. \quad (24)$$

This system was reduced to

$$2b=a'+\alpha \quad (25)$$

$$c=c_0+c_1x \quad (26)$$

$$g=g_2a^{-3}\exp\left(-\int\frac{\alpha\,dx}{a}\right), \quad (27)$$

$$f=a^{-5/2}\exp\left(-\int\frac{\alpha\,dx}{2a}\right)\left[f_2-3g_2\int ca^{-3/2}\exp\left(\int\frac{\alpha\,dx}{a}\right)dx\right], \quad (28)$$

where α, c_0, c_1, f_2 and g_2 are arbitrary constants. Setting

$$X=\int\frac{dx}{a} \quad (29)$$

$$Y=y\exp\left(-\int\frac{b\,dx}{a}\right)-\int\frac{c}{a}\exp\left(-\int\frac{b\,dx}{a}\right)dx, \quad (30)$$

(18) can be transformed into autonomous form. When $c\neq 0$, a satisfied the non-linear fourth order equation

$$caa''''+\left[c\left(\frac{5a'}{2}+\frac{\alpha}{2}\right)-c'a\right]a'''=-12g_2c^3a^{-3}\exp\left(-\int\frac{\alpha\,dx}{a}\right), \quad (31)$$

and (18) became

$$Y''+\alpha Y'+\left(M+\frac{\alpha^2}{4}\right)Y=g_2Y^3+f_2Y^2+N. \quad (32)$$

When $c=0$

$$a=a_0+a_1x+a_2x^2, \quad (33)$$

and (18) became

$$Y''+\alpha Y'+\beta Y=g_2Y^3+f_2Y^2, \quad (34)$$

where

$$\beta=-\frac{1}{4}a_1^2+a_0a_2+\frac{1}{4}\alpha^2, \quad (35)$$

and M, N, a_0, a_1 and a_2 are constants. The quantities M and N are constants of integration and are given by [15]

$$M=\frac{1}{2}aa''-\frac{1}{4}a'^2-2f_2I+3g_2I^2 \quad (36)$$

and

$$N=-a^{-1/2}\exp\left(-\int\frac{\alpha\,dx}{2a}\right)\left(ac'-\frac{1}{2}a'c+\frac{1}{2}\alpha c\right)-\left(\frac{1}{2}aa''-\frac{1}{4}a'^2+\frac{\alpha^2}{4}\right)I+f_2I^2-2g_2I^3, \quad (37)$$