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Pendulum with a square-wave modulated length

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ABSTRACT

Parametric excitation of a rigid planar pendulum caused by a square-wave modulation of its length is investigated both analytically and with the help of computer simulations. The threshold and other characteristics of parametric resonance are found and discussed in detail. The role of non-linear properties of the pendulum in restricting the resonant swinging is emphasized. The boundaries of parametric instability are determined as functions of the modulation depth and the quality factor. Stationary oscillations at these boundaries and at the threshold conditions are investigated. The feedback providing active optimal control of pumping and damping is analyzed. Phase locking between the drive and the pendulum at large amplitudes and the phenomenon of parametric autoresonance are discussed.

1. Introduction: the investigated physical system

Periodic excitation of a physical system is called *parametric forcing* if it is realized by variation of some parameter that characterizes the system. In particular, a pendulum can be excited parametrically by a given vertical motion of its suspension point. In the frame of reference associated with the pivot, such forcing of the pendulum is equivalent to periodic modulation of the gravitational field. This apparently simple physical system exhibits a surprisingly vast variety of possible regular and chaotic motions. Hundreds of texts and papers are devoted to investigation of the pendulum with vertically oscillating pivot: see, for example, Refs. [1,2] and references therein. A widely known curiosity in the behavior of an ordinary rigid planar pendulum whose pivot is forced to oscillate along the vertical line is the dynamic stabilization of its inverted position, occurring when the driving amplitude and frequency lie in certain intervals (see, for example, Refs. [2–5]).

Another familiar method of parametric excitation which we explore in this paper consists of a periodic variation of the length of the pendulum. In many textbooks and papers (see, for example, Refs. [6–11]) such a system is considered as a simple model of a playground swing. Indeed, the swing can be treated as a physical pendulum whose effective length changes periodically as the child squats at the extreme points, and straightens each time the swing passes through the equilibrium position.

A fascinating description of an exotic example illustrating this mode of parametric excitation can be found in Ref. [12, p. 27]. In Spain, in the cathedral of a northern town Santiago de Compostela, there is a famous *O Botafumeiro*, a very large incense burner suspended by a long rope, which can swing through a huge arc. The censer is pumped by periodically shortening and lengthening the rope as it is wound up and then down around the rollers supported high above the floor of the nave. The pumping action is carried out by a squad of priests, called tiraboleiros, each holding a rope that is a strand of the main rope that goes from the pendulum to the rollers and back down to near the floor. The tiraboleiros periodically pull on their respective ropes in response to orders from the chief verger of the cathedral. One of the more terrifying aspects of the pendulum's motion is the fact that the amplitude of its swing is very large, and it passes through the bottom of its arc with a high velocity, spewing smoke and flames.

In this paper we consider a pendulum with modulated length that can swing in the vertical plane in the uniform gravitational field. To allow arbitrarily large swinging and even full revolutions, we assume that the pendulum consists of a rigid massless rod (rather than a flexible string) with a massive small bob on its end. The effective length of the pendulum can be changed by shifting the bob up and down along this rod. Periodic modulation of the effective length by such mass redistribution can cause, under certain conditions, a growth of initially small natural oscillations. This phenomenon is a special case of parametric resonance.

2. The square-wave modulation of the pendulum length

In this paper we are concerned with a periodic square-wave (piecewise constant) modulation of the pendulum length.

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The square-wave modulation provides an alternative and may be more straightforward way to understand the underlying physics and to describe quantitatively the phenomenon of parametric resonance in comparison with a smooth (e.g., sinusoidal) modulation of the pendulum length [6–11]. A computer program [13] developed by the author simulates such a physical system and aids greatly in investigating the phenomenon.

In the case of the square-wave modulation, abrupt, almost instantaneous increments and decrements in the length of the pendulum occur sequentially, separated by equal time intervals. We denote these intervals by $T/2$, so that T equals the period of the length variation (the period of modulation). The square-wave modulation can produce considerable oscillation of the pendulum if the period and phase of modulation are chosen properly. For example, suppose that the bob is shifted up (toward the axis) at an instant at which the pendulum passes through the lower equilibrium position, when its angular velocity reaches a maximum value. While the weight is moved radially, the angular momentum of the pendulum with respect to the pivot remains constant. Thus the resulting reduction in the moment of inertia is accompanied by an increment in the angular velocity, and the pendulum gains additional energy. The greater the angular velocity, the greater the increment in energy. This additional energy is supplied by the source that moves the bob along the rod. On the other hand, if the bob is instantly moved down along the rod of the swinging pendulum, the angular velocity and the energy of the pendulum diminish. The decrease in energy is transferred back to the source.

In order that the modulation of the length regularly feeds the pendulum with energy, the period and phase of modulation must satisfy certain conditions. In particular, the most rapid growth of the amplitude occurs if effective length of the pendulum is reduced each time the pendulum crosses the equilibrium position, and is increased back near greatest elongations, when the angular velocity is almost zero. This radial displacement of the bob into its former position causes nearly no decrement in the kinetic energy. The resonant growth of the amplitude occurs if two cycles of modulation are executed during one period of natural oscillations. This is the principal parametric resonance. The time history of such oscillations for the case of a very weak friction (quality factor $Q=1500$) is shown in Fig. 1 together with the square-wave variation of the pendulum length. The graph of $\varphi(t)$ is obtained by numerical integration of the equations of motion, see Eqs. (3) and (4) below, by the 4th order Runge–Kutta method.

The growth of the amplitude at parametric resonance is restricted by non-linear effects. In a non-linear system like the pendulum, the natural period depends on the amplitude of oscillations. As the amplitude grows, the natural period of the pendulum becomes longer. However, in the accepted model the drive period (period of modulation) remains constant. If conditions for parametric excitation are fulfilled at small oscillations and the amplitude is growing, the conditions of resonance become violated at large amplitudes. The drive drifts out of phase with the pendulum and the phase relations change gradually to those favorable for the backward transfer of energy from the pendulum to the source of modulation. This causes gradual reduction of the amplitude. The natural period becomes shorter, and conditions for the growth of the amplitude restore. Oscillations acquire the character of beats, as shown in Fig. 1. Due to friction these transient beats gradually fade, and the amplitude tends to a finite constant value.

Details of the process of resonant growth followed by a non-linear restriction of the amplitude for parametrically excited pendulum ($T=T_0/2$) with considerable values of the modulation depth and friction are shown in Fig. 2. The vertical segments of the phase trajectory and of the $\dot{\varphi}(t)$ graph correspond to instantaneous increments and decrements of the angular velocity $\dot{\varphi}$ at the instants at which the bob is shifted up and down respectively. The curved portions of the phase trajectory that spiral in toward the origin correspond to damped natural motions of the pendulum between the jumps of the bob. The initially fast growth of the amplitude (described by the expanding part of the phase trajectory) gradually slows down, because the natural period becomes longer. After reaching the maximum value of 78.3°, the amplitude alternatively decreases and increases within a small range approaching slowly its final value of about 74°. The initially unwinding spiral of the phase trajectory simultaneously approaches the closed limit cycle, whose characteristic shape can be seen in the left-hand part of Fig. 2.

The energy of the pendulum can increase not only when two full cycles of variation in the parameter occur during one natural period, but also when two cycles occur during three, five or any odd number of natural periods (resonances of odd orders). The delivery of energy, though less efficient, is also possible if two cycles of modulation occur during an even number of natural periods (resonances of even orders). Generally, parametric resonance can occur when one of the following conditions for the frequency ω (or for the period T) of modulation is fulfilled:

$$\omega=\omega_n=\frac{2\omega_0}{n}, \quad T=T_n=\frac{nT_0}{2},\ n=1,2,\ldots \tag{1}$$

In conditions of parametric resonance both the investment in energy caused by the modulation of a parameter and the frictional losses are proportional to the energy stored. Therefore, parametric resonance is possible only when a *threshold* is exceeded, that is, when the increment in energy during a period (caused by the parameter variation) is larger than the amount of energy dissipated during the same time. The critical (threshold) value of the modulation depth depends on friction. However, if the threshold is exceeded, the frictional losses of energy cannot restrict the growth of the amplitude. With friction, stationary oscillations of a finite amplitude eventually establish due to non-linear properties of the pendulum.

We assume that the changes in the length l of the pendulum occur between $l_1=l_0(1+m_l)$ and $l_2=l_0(1-m_l)$, where m_l is the

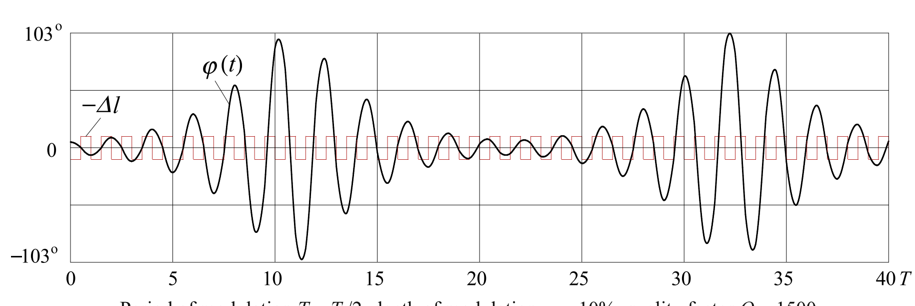

Fig. 1. Initial exponential growth of the amplitude of oscillations at parametric resonance of the first order ($n=1$) under the square-wave modulation, followed by beats.