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Pressure-driven flow of a rate-type fluid with stress threshold in an infinite channel

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A B S T R A C T

In this paper we extend some of our previous works on continua with stress threshold. In particular here we propose a mathematical model for a continuum which behaves as a non-linear upper convected Maxwell fluid if the stress is above a certain threshold and as a Oldroyd-B type fluid if the stress is below such a threshold. We derive the constitutive equations for each phase exploiting the theory of natural configurations (introduced by Rajagopal and co-workers) and the criterion of the maximization of the rate of dissipation. We state the mathematical problem for a one-dimensional flow driven by a constant pressure gradient and study two peculiar cases in which the velocity of the inner part of the fluid is spatially homogeneous.

1. Introduction

A large variety of materials such as food products, polymers, paints, oils and foams cannot be described by the classical linear viscous model. For this reason a large class of non-Newtonian models have been developed with the aim of explaining non-linear behaviors such as shear thinning/thickening, non-linear creep and stress relaxation.

In his celebrated work [7] Maxwell developed the first rate-type fluid model capable of describing stress relaxation, while later Burgers [1] developed a rate-type model for describing some geomaterials which included the classical rate-type model due to Oldroyd [8], namely the Oldroyd-B type model. Oldroyd was the first to develop a consistent framework for the rheology of rate-type viscoelastic fluids, focussing on the importance of the frame invariance and introducing some kinds of derivatives to obtain proper frame indifferent constitutive equations.

Since these seminal works, a plethora of models for viscoelastic response have been developed and numerous frame-invariant time derivatives have been introduced. Rajagopal and co-workers have developed in [10] a proper thermodynamical framework from which most of the viscoelastic constitutive relations can be derived.

The laminar flow of rate-type fluids have been extensively studied both in planar and cylindrical geometries. Waters and King [16] studied the pressure driven flow of an Oldroyd-B fluid in a straight cylindrical pipe, obtaining exact solutions by means of Laplace transform method. Rahaman and Ramkissoon [9] have studied the non-stationary flow of a Maxwell fluid in a pipe. Steady solutions due to oscillating cylindrical boundaries for second grade and Oldroyd-B type fluids have been obtained by Rajagopal [12] and Rajagopal and Bhatnagar [13].

In this paper we investigate the behavior of a non-Newtonian incompressible rate-type fluid which switches from an Oldroyd-B behavior to a non-linear Maxwell behavior depending on whether the stress is larger or smaller than a certain threshold. A typical example of a continuum that changes its behavior depending on the value of some function of the stress is the so-called Bingham fluid, which is a Newtonian viscous fluid that exhibits a threshold (the so-called yield stress) below which the strain rate is zero (so that no deformations occur).

In previous works we have studied a series of extensions of this simple model and we have investigated the corresponding mathematical problems in one-dimensional settings. The first extension was to the case in which the region where the stress is below the threshold behaves like a Neo-Hookean elastic solid (see [2]) and we have subsequently extended this case to the one in which the same region behaves like a visco-elastic Maxwell fluid [3].

We have then studied the case of an elastic material such that no deformation occurs above a certain threshold [4] and we have investigated the case in which the transition from rigid to elastic occurs when the stress becomes greater than the threshold (see [5,6]). The methodology developed in all these papers can be used to formulate a variety of models for continuum with stress threshold.

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All the models were obtained in the framework of the theory of natural configurations developed by Rajagopal and co-workers [11]. Depending on the constitutive equations for the phases constituting the continuum, we have obtained mathematical formulations of various complexity. In general these formulations consist of a free boundary problems involving hyperbolic and parabolic equations.

In this paper we study the case of a material in which one "phase" behaves as a non-linear Maxwell fluid while the other as a Jeffreys fluid. For the sake of simplicity we will choose for the latter the Oldroyd-B fluid model while for the former the upper convected Maxwell model. Minor changes allow to consider other models, like corotational Maxwell model or interpolated Maxwell model for the Maxwell fluid, and generalized Jeffreys models for the rate-type fluid (e.g. Oldroyd A).

The general constitutive equation for a non-linear Maxwell fluid is given by

$$\lambda \frac{D\mathbb{T}}{Dt} + \mathbb{T} = 2\eta\mathbb{D}, \tag{1}$$

where λ is a positive parameter, $\mathbb{T}$ is the stress tensor, η is the viscosity and $\mathbb{D}$ is the symmetric part of the velocity gradient, and where $D(\cdot)/Dt$ stands for any frame-invariant derivative (upper convected, lower convected, corotational). On the other hand, the general constitutive equation for a generalized Jeffreys model is

$$\lambda_1 \frac{D\mathbb{T}}{Dt} + \mathbb{T} = 2\eta\left[\mathbb{D} + \lambda_2 \frac{D\mathbb{D}}{Dt}\right], \tag{2}$$

λ_1 and λ_2 being positive parameters. In both cases the stress is obtained by solving a differential equation involving a symmetric tensor (meaning six scalar differential equations). In the framework of the theory of natural configurations we will obtain specific forms for Eqs. (1) and (2) imposing the elastic response of the continuum, the way the body dissipates energy and requiring that the dissipation is maximum.

As we said we will focus on two specific kinds of non-linear visco-elastic fluids, but the procedure we are going to describe can be applied to any fluid whose constitutive equations are of the type (1), (2).

Throughout the paper we use the terminology "fluid with a stress threshold", which seems to represent an oxymoron if one think to the very definition of a fluid. By the way, with this expression we intend to indicate that the switching from the Oldroyd-B behavior to the Maxwell behavior occurs when the second invariant of the stress tensor reaches a fixed threshold value.

2. Kinematical results

We consider a material domain $\Omega \in \mathcal{R}^3$ and we assume that there exists a surface Γ that divides the domain into two regions Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$. We suppose that the material that occupies the region Ω_1 behaves as a upper convected Maxwell fluid, while the material within the region Ω_2 as an Oldroyd-B fluid. We assume that Ω evolves in time so that at time $t>0$ the domain is $\Omega_t = \Omega_{1t} \cup \Omega_{2t}$, we define Γ_t as the sharp interface separating the domains Ω_{1t} and Ω_{2t} (see Fig. 1). The motion of a particle $\vec{x} \in \Omega$ is given by

$$\vec{x} = \Theta(\vec{X}, t), \tag{3}$$

where $\vec{X} \in \Omega$ is the Lagrangian coordinate and $\vec{x}$ is the Eulerian coordinate, that is the position of particle $\vec{X}$ at time t. We assume that Θ is invertible. The velocity, in the Eulerian coordinate system, is given by

$$\vec{v}(\vec{x}, t) = \left.\frac{\partial \Theta(\vec{X}, t)}{\partial t}\right|_{\vec{X} = \Theta^{-1}(\vec{x}, t)}, \tag{4}$$

and the acceleration by

$$\vec{a}(\vec{x}, t) = \left.\frac{\partial^2 \Theta(\vec{X}, t)}{\partial t^2}\right|_{\vec{X} = \Theta^{-1}(\vec{x}, t)}. \tag{5}$$

The deformation gradient of the motion is

$$\mathbb{F} =: grad\ \Theta(\vec{X}, t), \tag{6}$$

where *grad* denotes the gradient with respect to the Lagrangian coordinates. The velocity gradient is

$$\mathbb{L}(\vec{x}, t) =: \nabla \vec{v}(\vec{x}, t), \tag{7}$$

where ∇ is the gradient taken with respect to the Eulerian coordinates. The symmetric part of the velocity gradient is given by

$$\mathbb{D}(\vec{x}, t) =: \tfrac{1}{2}(\mathbb{L} + \mathbb{L}^T). \tag{8}$$

We introduce also the left and right Cauchy–Green tensors

$$\mathbb{B} = \mathbb{F}\mathbb{F}^T, \quad \mathbb{C} = \mathbb{F}^T\mathbb{F}. \tag{9}$$

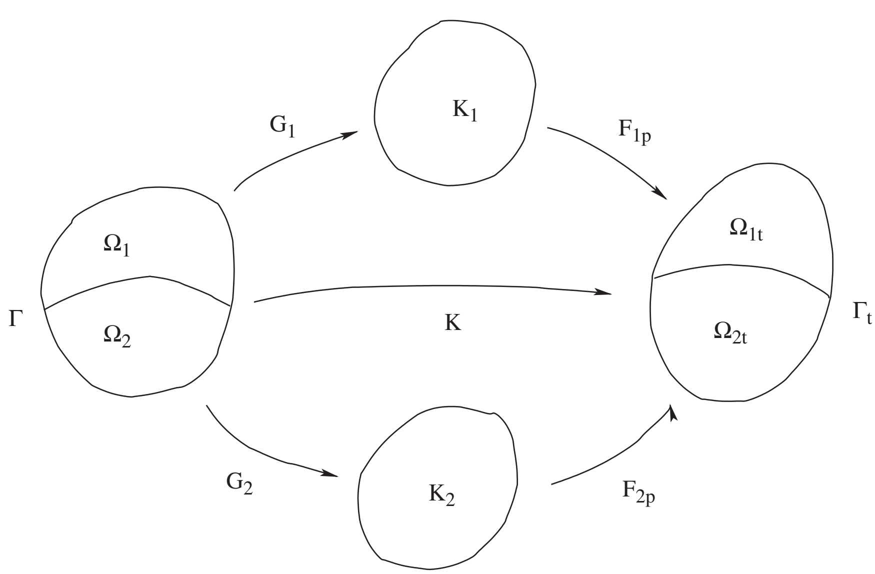

Fig. 1. Schematic diagram illustrating the evolution of the natural configurations of "phases" Ω_1 and Ω_2.