International Journal of Non-Linear Mechanics 46 (2011) 1001–1008

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

The Lie-group shooting method for boundary-layer problems with suction/injection/reverse flow conditions for power-law fluids

Chein-Shan Liu *

Department of Civil Engineering, National Taiwan University, Taipei, Taiwan

ARTICLE INFO

Article history:
Received 20 January 2010
Accepted 5 April 2011
Available online 15 April 2011

Keywords:
Blasius equation
Power-law fluids
Suction–injection boundary conditions
Reverse flow
Multiple-solutions
Lie-group shooting method

ABSTRACT

For power-law fluids we propose a Lie-group shooting method to tackle the boundary-layer problems under a suction/injection as well as a reverse flow boundary conditions. The Crocco transformation is employed to reduce the third-order equation to a second-order ordinary differential equation, and then through a symmetric extension to a boundary value problem with *constant boundary conditions*, which can be solved numerically by the Lie-group shooting method. However, the resulting equation is singular, which might be difficult to solve, and we propose a technique to overcome the initial impulse caused by the singularity using a very small time-step integration at the first few time steps. Because we can express the missing initial condition through a closed-form formula in terms of the weighting factor $r \in (0,1)$, the Lie-group shooting method is very effective for searching the multiple-solutions of a reverse flow boundary condition.

1. Introduction

The boundary-layer theory explains very well the steady-state flow over a flat plate at zero incidence angle known as the Blasius flow. It is long known that the appearance of boundary layers is not restricted to the canonical problem of the motion of a body through a viscous fluid. Several other technologically important sources of boundary-layer behavior are the flows behind expansion and shock waves travelling over smooth surfaces and the flow above a moving conveyor belt.

A main reason for the interest in analysis of boundary-layer flows over solid surfaces is the possibility by applying the theory to the efficient design of supersonic and hypersonic flights. Besides, the mathematical model considered in the present research has importance in studying many problems of engineering, meteorology, and oceanography; see, for example, Schlichting [1], Ozisik [2], Nachman and Callegari [3], Shu and Wilks [4], Hopwell [5], Zheng et al. [6,7], Zheng and Deng [8], Zheng and He [9], Zheng and Zhang [10].

Klemp and Acrivos [11,12] have studied boundary-layer flow caused by a finite flat plate that moves in the direction opposite to the main stream at high Reynolds numbers. They thought that a region of reverse flow remains confined within a boundary-layer and the conventional boundary-layer equations should continue to apply downstream of the point of detachment of the surface streamline. In the upstream portion of the separated region, the boundary-layer equation possesses a similarity solution. Hussaini and Lakin [13] showed that the solutions of such boundary-layer problems exist only up to a certain value of the velocity ratio parameter.

We assume that the moving flat plate is semi-infinite with a porous surface and that the plate is moving at a constant speed U_w in the direction parallel to an oncoming flow with a constant speed U_∞. By the assumption of incompressibility and the conservation of momentum, the laminar flow satisfies

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0, \tag{1}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=\frac{1}{\rho}\frac{\partial \tau_{XY}}{\partial Y}. \tag{2}$$

In above, X and Y are the coordinates attached to the plate in the horizontal and perpendicular directions, and U and V are respectively the velocity components of the flow in the X and Y directions. The fluid density ρ is assumed to be a constant.

The shear stress is governed by a power law:

$$\tau_{XY}=K\left|\frac{\partial U}{\partial Y}\right|^{N-1}\frac{\partial U}{\partial Y}, \tag{3}$$

where $K>0$ is a constant and the power $N>0$ reflects the discrepancy to the Newtonian fluids (with $N=1$), where in the case that $N<1$ is the power law of pseudo-plastic fluids and $N>1$ is the dilatant fluids. The corresponding boundary conditions are

* Tel.: +886 2 33664233; fax: +886 27396752.
E-mail address: liucs@ntu.edu.tw

0020-7462/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2011.04.016

given by

$$U(X,0)=U_w,\quad U(X,+\infty)=U_\infty,\quad V(X,0)=V_w(X)=V_0X^{-N/(N+1)}. \tag{4}$$

After introducing a similarity variable and a stream function:

$$\eta=BX^\beta Y,\quad \phi(X,Y)=AX^\sigma f(\eta) \tag{5}$$

with

$$\sigma=\frac{1}{N+1},\quad \beta=-\sigma,\quad B=\left(\frac{\rho U_\infty^{2-N}}{(N+1)K}\right)^{1/(N+1)},\quad A=\frac{U_\infty}{B},$$

we can obtain

$$(|f''(\eta)|^{N-1}f''(\eta))'+f(\eta)f''(\eta)=0, \tag{6}$$

which is subjected to the boundary conditions:

$$f(0)=-C,\quad f'(0)=\xi,\quad f'(+\infty)=1. \tag{7}$$

In above, $\xi=U_w/U_\infty$ is the velocity ratio. When $\xi<0$, we have a reverse flow attached near the boundary. When $0<\xi<1$, the speed of the oncoming fluid is larger than that of the plate. When $\xi>1$, the speed of the moving plate is faster than the speed of the oncoming fluid. The term $C=(N+1)BV_0/U_\infty$ is a constant related to the situation of suction if it is negative or injection if it is positive.

When $N=1$, the Blasius equation is recovered. Hussaini et al. [14] have used the similarity technique and the Crocco transformation to study the reverse flow solution for the Blasius equation under $C=0$ and $\xi<0$. They showed that a boundary-layer solution exists only if the ratio of the plate velocity to the mainstream velocity is below a critical value ξ^*, and under that condition the solutions are not unique. Soewono et al. [15] further studied the effect of the suction–injection boundary condition on the above boundary-layer problem.

Power-law fluids have been called the Ostwald–deWaeles fluids and have been well examined because the constitutive equation for such a fluid not only gives a good expression for a large portion of non-Newtonian fluids but also encompasses a Newtonian fluid as well. The theoretical boundary-layer theory for power-law fluids was first investigated by Schowalter [16], and then Acrivos et al. [17] obtained a similarity solution. The experimental results that significant drag reduction can be achieved by injecting fluid into the boundary layer motivated the investigations of non-Newtonian boundary-layer flows with injection or suction at the surface. Flows with suction or injection through a porous wall are of practical interest for cooling, delaying transition to turbulence, and prevention of separation in an adverse pressure gradient.

Since the effects of suction (or injection) on the boundary-layer flow are of interest in increasing (or decreasing) the drag force and in controlling the boundary separation, the purpose of this paper is to investigate the behavior for a reverse flow boundary-layer flow caused by a flat plate moving opposite to the stream in a power-law fluid with suction or injection boundary condition. Previously, we have applied the Lie-group shooting method to tackle the boundary-layer problems with certain performance [18–20]. This paper is an extension of these works.

This paper is organized as follows. In Section 2 we adopt the Crocco transformation to reduce Eq. (6) into a *non-linear singular* second-order ODE, and arrange it into another second-order ODE with *constant boundary conditions* by a *symmetric extension*. In Section 3 we introduce some mathematical requirements of the Lie-group formulations of the resulting ODEs. Then, a Lie-group shooting method is constructed in Section 4. Section 5 specializes the obtained second-order ODE, and develops a closed-form formula to calculate the unknown initial conditions. Numerical examples and calculations are given in Section 6, and finally we draw conclusions in Section 7.

2. The Crocco transformation

The boundary-layer equations are encountered in many engineering applications, such as airfoil, liquid transport by belt conveyor, and many others. Since the 1960s the researchers working on this problem have been using the Crocco transformation in which the tangential velocity f' becomes a new independent variable by setting

$$z=f'(\eta), \tag{8}$$

while the new dependent variable is the shear force

$$g(z)=[f''(\eta)]^N. \tag{9}$$

In this paper we propose a new method for the computation of the following second-order boundary-layer equation:

$$g^{1/N}g''=-z,\quad \xi<z<1, \tag{10}$$

$$g'(\xi)=C,\quad g(1)=0, \tag{11}$$

which are obtained by the above transformations in Eqs. (8) and (9) applied to the boundary-layer equations (6) and (7). If g is allowed to be negative, the term $g^{1/N}$ in Eq. (10) can be replaced by $|g|^{1/N}$.

By letting

$$z=(1-\xi)t+\xi, \tag{12}$$

$$y(t)=g(z)-(1-\xi)Ct+C_0, \tag{13}$$

we can transform Eqs. (10) and (11) into an equivalent system:

$$\ddot{y}=-\frac{(1-\xi)^2\xi+(1-\xi)^3t}{|y+(1-\xi)Ct-C_0|^{1/N}},\quad 0<t<1, \tag{14}$$

$$\dot{y}(0)=0,\quad y(1)=c, \tag{15}$$

where $c=C_0-(1-\xi)C$. The superimposed dot denotes the differential with respect to t. Here, C_0 is a translation constant being selected such that $c>0$.

Through a symmetric extension of Eq. (14) defined in the interval of $t\in(0,1)$ into the interval of $t\in(-1,0)$, we can write the governing equation to be

$$\ddot{y}=f(t,y),\quad -1<t<1, \tag{16}$$

$$y(-1)=y(1)=c, \tag{17}$$

where

$$f(t,y)=-\frac{(1-\xi)^2\xi+(1-\xi)^3|t|}{|y+(1-\xi)C|t|-C_0|^{1/N}}. \tag{18}$$

Because of the symmetric property of Eq. (16), if the condition $\dot{y}(0)=0$ can be guaranteed to be satisfied then the second condition in Eq. (17) is fulfilled automatically. In Section 5 we can see what advantage can be gained by adjusting the two boundary conditions in Eq. (17) being equal.

3. One-step group-preserving scheme

3.1. The group-preserving scheme

Let us write Eq. (16) in a vector form

$$\dot{\mathbf{y}}=\mathbf{f}(t,\mathbf{y}), \tag{19}$$

where

$$\mathbf{y}:=\begin{bmatrix}y_1\\ y_2\end{bmatrix},\quad \mathbf{f}:=\begin{bmatrix}y_2\\ f(t,y_1)\end{bmatrix}. \tag{20}$$