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Role of surface effects in the finite deformation of an elastic solid with elliptical hole

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ABSTRACT

We study the finite plane deformations of a particular harmonic material surrounding an elliptical hole whose boundary incorporates the contribution of surface mechanics. We are particularly interested in the distribution of the Piola hoop stress along the edge of the hole. Surprisingly, in the absence of any external loading, the hoop stress induced solely by the surface effects is identical to that in the corresponding case in a *linearly* elastic solid. In addition, we show that even in the presence of surface effects, we can nevertheless design a so-called 'harmonic hole' where the Piola mean stress remains constant everywhere in the surrounding solid. In this case, however, the hoop stress is no longer constant along the edge of the elliptical hole due to the contribution from surface energy.

1. Introduction

Micromechanical analysis of composites is fundamental to a better understanding of the local and overall behavior of composite materials. In structures analyzed at the nanoscale, the surface to volume ratio becomes sufficiently high to require the incorporation of surface energy into any continuum model of deformation. Gurtin and co-workers [1,2] first proposed a surface elasticity model to account for the contribution from surface energies. The theory behind this model was further improved and clarified by Ru in [3]. Recently, variations of the surface elasticity model have been incorporated into several interesting and challenging problems in micromechanics including the analysis of nano-composites containing Eshelby's inclusions, inhomogeneities, cavities and cracks [4–10]. In all of these previous studies, the bulk material has been assumed to be linearly elastic mainly in an effort to avoid the formidable analytical challenges presented by the theory of non-linear elasticity.

Fortunately, an elegant and powerful complex variable formulation for plain-strain finite deformations of a particular set of compressible hyperelastic materials of harmonic type has been developed by Ru [11]. This new complex variable formulation has been employed successfully to study various crack and inclusion problems in harmonic solids [11–14] undergoing finite deformations. In this paper, we adopt Ru's formulation and present the analysis of the contribution of surface energy to a bulk harmonic solid surrounding an elliptical nano-hole subjected to finite deformation.

2. Basic formulation

2.1. Complex variable formulation for a harmonic solid

Let the complex variable $z=x_1+ix_2$ be the initial coordinates of a material particle in the undeformed configuration and $w(z)=y_1(z)+iy_2(z)$ the corresponding spatial coordinates in the deformed configuration. Define the deformation gradient tensor as having components

$$F_{ij}=\frac{\partial y_i}{\partial x_j}. \tag{1}$$

For a particular class of harmonic materials, the strain energy density W defined with respect to the undeformed unit area can be expressed by

$$W=2\mu[F(I)-J],\quad F'(I)=1/4\alpha[I+\sqrt{I^2-16\alpha\beta}]. \tag{2}$$

Here I and J are the scalar invariants of FF^T given by

$$I=\lambda_1+\lambda_2=\sqrt{F_{ij}F_{ij}+2J},\quad J=\lambda_1\lambda_2=\det[F_{ij}], \tag{3}$$

where λ_1 and λ_2 are the principal stretches, μ is the shear modulus and $1/2\le\alpha<1$, $\beta>0$ are the two material constants.

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Despite its limitations (for example, this model is not suitable in cases when the solid is subjected to significant compression), this special class of harmonic materials has attracted considerable attention in the literature [12–19].

According to the formulation developed recently by Ru [11], the deformation w can be written in terms of two analytic functions $\varphi(z)$ and $\psi(z)$ as

$$\mathrm{i}w(z,\bar{z}) = \alpha\varphi(z) + \mathrm{i}\overline{\psi(z)} + \frac{\beta z}{\overline{\varphi'(z)}}, \tag{4}$$

and the complex Piola stress function Φ is given by

$$\Phi(z,\bar{z}) = 2\mathrm{i}\mu\left[(\alpha-1)\varphi(z) + \mathrm{i}\overline{\psi(z)} + \frac{\beta z}{\overline{\varphi'(z)}}\right]. \tag{5}$$

In addition, the Piola stress components can be written in terms of the Piola stress function Φ as

$$-\sigma_{21} + \mathrm{i}\sigma_{11} = \Phi_{,2}, \quad \sigma_{22} - \mathrm{i}\sigma_{12} = \Phi_{,1}. \tag{6}$$

2.2. Surface elasticity theory

Using the concept of surface stress, the boundary condition on the surface is given by [1–3,10]

$$\sigma_{\alpha j} n_j \underline{e}_\alpha + \sigma^s_{\alpha\beta,\beta}\underline{e}_\alpha = 0 \quad \text{(tangential direction)}$$
$$\sigma_{ij} n_i n_j = \sigma^s_{\alpha\beta}\kappa_{\alpha\beta} \quad \text{(normal direction)} \tag{7}$$

where n_i is the unit normal vector of the surface, $\sigma^s_{\alpha\beta}$ is the (2×2) symmetric surface stress tensor, and $\kappa_{\alpha\beta}$ is the curvature tensor of the surface.

In the theory of surface elasticity [6,8,9], the surface stress tensor $\sigma^s_{\alpha\beta}$ is related to the deformation dependent surface energy γ. In this study, we proceed as in [6,8,9] and consider only the simple case in which the surface energy is independent of the deformation. Consequently the surface stress tensor can be simply expressed in terms of the surface energy as

$$\sigma^s_{\alpha\beta} = \gamma\delta_{\alpha\beta}, \tag{8}$$

which implies that the surface is isotropic [6,8].

3. An elliptical hole with surface energy in a harmonic solid

Consider a harmonic solid containing an elliptical hole whose boundary is assumed to incorporate surface effects through the surface elasticity theory described above. We further assume that the solid is subjected to a uniform Piola stress field $(\sigma^\infty_{11}, \sigma^\infty_{22}, \sigma^\infty_{12}, \sigma^\infty_{21})$ at infinity. The boundary of the elliptical hole is described by $\Gamma: \{x_1^2/a^2 + x_2^2/b^2 = 1\}$ where a and b are, respectively, the semi-major and semi-minor axes. We introduce the following conformal mapping function:

$$z = x_1 + \mathrm{i}x_2 = \omega(\xi) = R\left(\xi + \frac{m}{\xi}\right), \tag{9}$$

where

$$R = \frac{a+b}{2}, \quad m = \frac{a-b}{a+b}. \tag{10}$$

This mapping maps the unbounded region outside the elliptical hole in the z-plane onto the exterior of unit circle $|\xi| \geq 1$ in the ξ-plane.

It follows from Eqs. (7) and (8) that the boundary condition on the surface of the elliptical hole with surface energy can be written explicitly as

$$\sigma_{nn} + \mathrm{i}\sigma_{nt} = \frac{\gamma}{\rho}, \tag{11}$$

where the subscripts n and t denote the normal and tangential directions of the elliptical boundary, respectively, and ρ denotes the radius of curvature of the elliptical hole and is given by ([9])

$$\rho = R\left(1 - \frac{m}{\xi^2}\right)^{3/2}(1 - m\xi^2)^{3/2}(1-m^2)^{-1} \quad \text{on} \quad \xi = e^{\mathrm{i}\theta} \tag{12}$$

The boundary condition (11) can be written in terms of the two analytic functions $\varphi(\xi) = \varphi(\omega(\xi))$ and $\psi(\xi) = \psi(\omega(\xi))$ as

$$2\mathrm{i}\mu\left[(\alpha-1)\varphi(\xi) + \mathrm{i}\overline{\psi(\xi)} + \frac{\beta\omega(\xi)\overline{\omega'(\xi)}}{\overline{\varphi'(\xi)}}\right] = \Phi_\Gamma(\xi) \quad \text{on} \quad \xi = e^{\mathrm{i}\theta} \tag{13}$$

where

$$\Phi_\Gamma(\xi) = F_0(\xi) = \gamma\sqrt{\frac{\xi^2 - m}{1 - m\xi^2}}, \tag{14}$$

which indicates that $\Phi_\Gamma(\xi) \equiv F_0(\xi)$ defined in Eq. (17) by Wang and Wang [9].

Using the method of analytic continuation, Eq. (13) can be solved for the two analytic functions $\varphi(\xi)$ and $\psi(\xi)$ as follows:

$$\varphi(\xi) = \frac{\mathrm{i}(\bar{B}R + (\beta R m/\bar{A}))}{1-\alpha}\xi^{-1} + \mathrm{i}AR\xi,$$

$$\psi(\xi) = -\frac{\gamma}{2\mu}\sqrt{\frac{1-m\xi^2}{\xi^2-m}} + \bar{A}R(1-\alpha)\xi^{-1}$$
$$-\frac{[A\beta R(1-\alpha)(1-m^2) + \beta R m(\bar{B} + (\beta m/\bar{A}))]\xi^2 - A\beta R m(1-\alpha)}{A^2(1-\alpha)\xi^3 - A(\bar{B} + (\beta m/\bar{A}))\xi} + BR\xi, \tag{15}$$

where A and B are complex constants determined by the remote uniform Piola stresses $(\sigma^\infty_{11}, \sigma^\infty_{22}, \sigma^\infty_{12}, \sigma^\infty_{21})$ such that

$$(1-\alpha)A - \frac{\beta}{\bar{A}} = \frac{\sigma^\infty_{11} + \sigma^\infty_{22} + i(\sigma^\infty_{21} - \sigma^\infty_{12})}{4\mu}, \quad B = \frac{\sigma^\infty_{11} - \sigma^\infty_{22} - i(\sigma^\infty_{12} + \sigma^\infty_{21})}{4\mu}. \tag{16}$$

In view of the fact that the Piola mean stress is determined by

$$\sigma_{11} + \sigma_{22} = 4\mu\,\mathrm{Im}\left\{(1-\alpha)\frac{\varphi'(\xi)}{\omega'(\xi)} + \frac{\beta\overline{\omega'(\xi)}}{\overline{\varphi'(\xi)}}\right\}, \tag{17}$$

the surface energy does not contribute to the distribution of the Piola mean stress in the harmonic solid. In other words, the Piola mean stress in the harmonic solid is solely induced by the applied remote uniform Piola stresses.

In addition, in order to ensure that $F'(I) \neq 0$ outside the elliptical hole, we must have that [12]

$$A(1-\alpha)\xi^2 \neq \bar{B} + \frac{\beta m}{\bar{A}} \quad (|\xi| \geq 1) \tag{18}$$

Consequently the following inequality must hold for the two loading parameters A and B:

$$(1-\alpha)|A| > \left|B + \frac{\beta m}{A}\right|. \tag{19}$$

Remark. $F'(I) \neq 0$ guarantees strong ellipticity of the associated system and is necessary in order to arrive at a general solution of the deformation $w(z,\bar{z})$ in Eq. (4) (see Ref. [11] for more details).

In the next section, several special cases will be discussed to more clearly illustrate the general solution obtained above.