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Using proper orthogonal decomposition to model off-reference flow conditions

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ABSTRACT

Proper orthogonal decomposition results from subsonic and transonic flow regimes are presented, and the full-order model is compared with the reduced-order models. The reduced-order models use basis functions generated for on- and off-reference flow conditions. For on-reference flow conditions, proper orthogonal decomposition of the full-order model is performed to generate the basis functions. For off-reference flow conditions, basis functions are obtained through interpolating among basis functions corresponding to bracketing flow conditions. Interpolation is performed on a tangent space to a Grassmann manifold. This paper evaluates the accuracy of POD at off-reference flow conditions for subsonic and transonic flow regimes. The results show that interpolation yields good results for the subsonic cases, but the accuracy of the transonic cases is considerably lower. Furthermore, the energy spectrum is used to assess the necessary number of basis functions. It is demonstrated that in order to determine the number of basis functions, it is better to assess the variation of individual energy values, as opposed to the cumulative energy values.

1. Introduction

Despite continuous advances in computational power, the scope of high-fidelity computational fluid dynamic (CFD) results remains limited for applications requiring numerous repetitions. Examples of such applications include parametric studies and design iterations. This limited scope is particularly evident in computational aeroelasticity, for which the costs associated with unsteadiness of the flow and temporal variation of the mesh can be a computational burden.

The computational cost of a CFD simulation can be drastically reduced while providing high-fidelity results using reduced-order models (ROMs). Through model reduction, dominant spatial modes are used to describe the flow. The non-linear partial differential equations can then be reduced to ordinary differential equations from which the time coefficients that weight the spatial modes are calculated. ROMs are often employed in structural dynamics as well [1].

Proper orthogonal decomposition (POD) is a method through which snapshots of the flow obtained from the full-order model (FOM) are used to extract the optimal set of spatially dependent basis functions [2]. The large set of partial differential equations is then projected onto the basis functions, resulting in a much smaller set of ordinary differential equations.

POD-based ROMs have been reviewed in [3–5]. The effects of deforming meshes [6–8] and heaving airfoils [9] have been considered. More recently, POD-based ROMs have been developed for deformed wings [10].

Early POD-based ROMs focused on computing basis functions directly from snapshots of a FOM for the same flow parameters as the ROM [11]. This approach renders the computational savings of the ROM moot. For practical applications, it is necessary to extend the ROM to off-reference parameter sets [12].

Attempts to modify the POD basis functions to account for off-reference flow parameters can be broadly broken into three groups. The first approach is to directly interpolate between sets of basis functions. This approach is simple to implement but results in basis functions that are no longer orthogonal [13,14].

The second approach [15] augments the snapshot database with solutions from a variety of parameters. These snapshots are then used to compute POD basis functions directly. While this method produces orthogonal basis functions, it still relies on a single set of unchanging basis functions to span the entire parameter space.

The third approach is interpolation on a tangent space to a Grassmann manifold [14,16]. This method has been shown to produce good results over a wide range of parameters [14]. A special case of this method is subspace angle interpolation, which considers interpolation between only two sets of basis functions [13,17,18].

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In this paper, proper orthogonal decomposition results from subsonic and transonic flow regimes are presented, and the full-order model is compared with reduced-order models. The reduced-order models use basis functions generated for on- and off-reference flow conditions.

For on-reference flow conditions, proper orthogonal decomposition of the full-order model is performed to generate the basis functions. For off-reference flow conditions, basis functions are obtained through interpolating among basis functions corresponding to bracketing flow conditions. Interpolation is performed on a tangent space to a Grassmann manifold.

This paper offers several valuable insights. POD results are generated for several cases, and a comparison is made between subsonic and transonic flows. Additionally, the effects of interpolation order when using a Grassmann manifold are investigated for the different flow regimes. Furthermore, the energy spectrum is used to assess the necessary number of basis functions. It is demonstrated that in order to determine the number of basis functions, it is better to assess the variation of individual energy values, as opposed to the cumulative energy values.

2. Proper orthogonal decomposition

Proper orthogonal decomposition is a method through which an optimal set of orthogonal spatial basis functions is extracted from a set of data [2]. Typically, the mean is subtracted, and the difference between the variable, $\mathbf{U}(\mathbf{x},t)$, and the time average, $\overline{\mathbf{U}}(\mathbf{x})$, is approximated by a linear combination of the spatially dependent basis functions, which are weighted by time-dependent coefficients:

$$\tilde{\mathbf{U}}(\mathbf{x},t)\equiv\mathbf{U}(\mathbf{x},t)-\overline{\mathbf{U}}(\mathbf{x})\approx\sum_{j=1}^{m}a_j(t)\boldsymbol{\varphi}_j(\mathbf{x}). \tag{1}$$

$a_j(t)=(\tilde{\mathbf{U}}(\mathbf{x},t),\boldsymbol{\varphi}_j(\mathbf{x}))/(\boldsymbol{\varphi}_j(\mathbf{x}),\boldsymbol{\varphi}_j(\mathbf{x}))$, and $(\cdot,\cdot)$ is the inner product. The basis functions, $\boldsymbol{\varphi}_j$, span a subspace of finite dimension m. Through reduced-order modeling, the partial differential equations are projected onto the set of basis functions and reduced to a system of ordinary differential equations.

2.1. Optimality

The basis functions have been presumed mutually orthogonal to more efficiently span the subspace. $\tilde{\mathbf{U}}$ is equal to the sum of the approximation onto the basis and the error:

$$\tilde{\mathbf{U}}=\sum_{j=1}^{m}\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi}_j)}{(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)}\boldsymbol{\varphi}_j+\left(\tilde{\mathbf{U}}-\sum_{j=1}^{m}\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi}_j)}{(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)}\boldsymbol{\varphi}_j\right).$$

Since the error is orthogonal to the approximation, the Pythagorean theorem holds, and

$$\|\tilde{\mathbf{U}}\|^2=\left\|\sum_{j=1}^{m}\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi}_j)}{(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)}\boldsymbol{\varphi}_j\right\|^2+\left\|\tilde{\mathbf{U}}-\sum_{j=1}^{m}\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi}_j)}{(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)}\boldsymbol{\varphi}_j\right\|^2,$$

where $\|\cdot\|$ is the L^2-norm. Consequently, minimizing the time-averaged error is equivalent to maximizing the time-averaged approximation. Due to the orthogonality assumption, the time-averaged square of the norm of the approximation can be simplified to

$$\left\langle\left\|\sum_{j=1}^{m}\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi}_j)}{(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)}\boldsymbol{\varphi}_j\right\|^2\right\rangle=\left\langle\sum_{j=1}^{m}\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi}_j)^2}{(\boldsymbol{\varphi}_j,\boldsymbol{\varphi}_j)}\right\rangle,$$

where $\langle\cdot\rangle$ denotes the time average.

The norm of the approximation is maximized by determining the optimal basis functions that maximize the functional

$$J[\boldsymbol{\varphi}]\equiv\left\langle\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi})^2}{(\boldsymbol{\varphi},\boldsymbol{\varphi})}\right\rangle, \tag{2}$$

where the subscript j has been removed for convenience. Eq. (2) is extremized when

$$\frac{\partial J}{\partial\delta}[\boldsymbol{\varphi}+\delta\boldsymbol{\psi}]\Big|_{\delta=0}=0=2\left\langle\boldsymbol{\psi}^T\hat{\mathbf{A}}(t)\boldsymbol{\varphi}-\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi})^2}{(\boldsymbol{\varphi},\boldsymbol{\varphi})}(\boldsymbol{\psi},\boldsymbol{\varphi})\right\rangle, \tag{3}$$

where $\hat{\mathbf{A}}(t)\equiv\tilde{\mathbf{U}}(\mathbf{x},t)\otimes\tilde{\mathbf{U}}(\mathbf{x},t)$. The outer product is defined such that $(\tilde{\mathbf{U}}\otimes\tilde{\mathbf{U}})\boldsymbol{\varphi}\equiv(\boldsymbol{\varphi},\tilde{\mathbf{U}})\tilde{\mathbf{U}}$. Eq. (3) must hold for an arbitrary $\boldsymbol{\psi}$; therefore,

$$\left\langle\hat{\mathbf{A}}(t)\boldsymbol{\varphi}-\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi})^2}{(\boldsymbol{\varphi},\boldsymbol{\varphi})}\boldsymbol{\varphi}\right\rangle=\mathbf{0}.$$

Additionally, for a non-trivial solution for $\boldsymbol{\varphi}$, it is necessary that

$$\left|\left\langle\hat{\mathbf{A}}(t)-\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi})^2}{(\boldsymbol{\varphi},\boldsymbol{\varphi})}\mathbf{I}\right\rangle\right|=0. \tag{4}$$

Eq. (4) can be written equivalently as an eigenvalue problem,

$$\langle\hat{\mathbf{A}}\rangle\boldsymbol{\varphi}=\lambda\boldsymbol{\varphi}, \tag{5}$$

where the eigenvalue

$$\lambda=\left\langle\frac{(\tilde{\mathbf{U}},\boldsymbol{\varphi})^2}{(\boldsymbol{\varphi},\boldsymbol{\varphi})}\right\rangle. \tag{6}$$

If $\boldsymbol{\varphi}$ is normalized so that $\|\boldsymbol{\varphi}\|=1$, $\lambda=\langle a(t)^2\rangle$. Consequently, the eigenvectors with the largest eigenvalues are the most significant basis functions. Additionally, since $\hat{\mathbf{A}}$ is symmetric positive semi-definite, the eigenvectors are orthogonal.

2.2. Problem reduction

Assuming the number of snapshots, M, is less than the number of unknown values in each snapshot, n, the eigenvalue problem can be further simplified using the method of snapshots [19]. The basis functions are expressed as linear combinations of the snapshots of $\tilde{\mathbf{U}}$:

$$\boldsymbol{\varphi}_j(\mathbf{x})=\sum_{i=1}^{M}c_j(t_i)\tilde{\mathbf{U}}(\mathbf{x},t_i). \tag{7}$$

Substituting (7) into (5) yields a smaller eigenvalue problem

$$\mathbf{B}\mathbf{c}_j=\lambda_j\mathbf{c}_j,$$

where

$$[\mathbf{B}]_{i,k}\equiv\frac{1}{M}(\tilde{\mathbf{U}}(\mathbf{x},t_k),\tilde{\mathbf{U}}(\mathbf{x},t_i)),\quad\{\mathbf{c}_j\}_i\equiv c_j(t_i).$$

The size of the eigenvalue problem is thus reduced from $n\times n$ to $M\times M$.

2.3. Basis function interpolation

The ROM requires spatial functions, which have thus far been the basis functions obtained from applying POD to the snapshots generated by the FOM. Running the FOM for every ROM case is counter to the motivation behind ROMs. Therefore, as an alternative, functions can be generated by interpolating between basis functions corresponding to flow parameters that bracket the conditions of interest. In this paper, the basis functions for off-reference conditions are generated through interpolation on a tangent space to a Grassmann manifold [14,16]. The resulting basis functions are orthogonal.

For a set of L simulations corresponding to different flow conditions parameterized by χ, a set of basis functions,