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On the periodic solutions of a rigid dumbbell satellite placed at $\mathcal{L}_4$ of the restricted three body problem

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ABSTRACT

In this work we provide sufficient conditions for the existence of periodic solutions of a rigid dumbbell satellite placed in the equilateral equilibrium $\mathcal{L}_4$ of the Restricted Three Body Problem via Averaging Theory.

1. Introduction and statement of the main results

The general study of the dynamics of dumbbell-type satellites has been presented extensively in a vast literature. Eulerian, Lagrangian and Hamiltonian formulations of such dynamics have been the main tools used in the formulation of these problems. It is known that a dumbbell satellite is modeled by two point masses connected by a massless rod. The dumbbell is the lightest body as having a structure which then enables us to consider its rotation. Among the various aspects related to these problems that are discussed in the literature, we can highlight the following:

1. Equilibria and stabilities in dumbbell-type satellites, see for example [1,5,7,10,14].
2. Periodic solutions and bifurcations [3,4,6,9,11,15].

These problems are undoubtedly appealing in the field of astronautics because of the need for placing satellites in stable orbits with stable orientations and the dumbbell-type satellites is a first model for ulterior studies with more complicated systems.

In this work we consider a rigid dumbbell satellite formed by two points of masses M_1 and M_2 connected by a segment of constant length l placed at the equilateral point $\mathcal{L}_4$ of the Restricted Three Body Problem, see [13] for more details about the RTBP. Following the methods exposed in Appendix B of the present paper, the equations of motion governing the attitude dynamics of the rigid dumbbell satellite are

$$\frac{d^2\theta}{dt^2}-\left(1+\frac{d\phi}{dt}\right)^2\sin\theta\cos\theta=\frac{3}{4}\sin\theta\cos\theta(2-\cos 2\phi+2\sqrt{3}\mu\sin 2\phi),$$
$$\frac{d^2\phi}{dt^2}\sin\theta+2\frac{d\theta}{dt}\left(1+\frac{d\phi}{dt}\right)\cos\theta=\frac{3}{4}\sin\theta(\sin 2\phi+2\sqrt{3}\mu\cos 2\phi) \quad (1)$$

with θ and ϕ the eulerian angles of nutation and precession and $\mu\in(0,1/2)$ the reduced mass of the Restricted Three Body Problem.

In particular $\theta=0$ is a position of instability for all values of ϕ. A stationary position is given by $\theta=\pi/2$ and $\phi=\phi_0$ where ϕ_0 is a solution of the equation:

$$\sin 2\phi_0+\sqrt{12}\mu\cos 2\phi_0=0.$$

This equation is satisfied by two values of ϕ_0 differing by $\pi/2$, the one position being stable, the other unstable.

We consider only the motion in the vicinity of the stable position, i.e. *we are only interested in small oscillations around this equilibrium*.

By means of the change of coordinates $x=\theta-\pi/2$ and $y=\phi-\phi_0$, linearizing Eqs. (1) in this equilibrium we obtain

$$\frac{d^2x}{dt^2}+a_\theta x=0,$$
$$\frac{d^2y}{dt^2}+a_\phi y=0 \quad (2)$$

with

$$a_\theta=\tfrac{3}{2}\sqrt{1+12\mu^2},\quad a_\phi=\tfrac{5}{2}+\tfrac{3}{4}\sqrt{1+12\mu^2}.$$

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The objective of this work is to provide a system of non-linear equations whose simple zeros provide periodic solutions of the perturbed dumbbell satellite with equations of motion:

$$\frac{d^2x}{dt^2}+a_\theta x=\varepsilon F_1\left(t,x,\frac{dx}{dt},y,\frac{dy}{dt}\right),$$
$$\frac{d^2y}{dt^2}+a_\phi y=\varepsilon F_2\left(t,x,\frac{dx}{dt},y,\frac{dy}{dt}\right), \qquad (3)$$

where ε is a small parameter. Here the smooth functions F_1 and F_2 define the perturbation. These functions are periodic in t and in resonance p:q with some of the periodic solutions of the unperturbed dumbbell satellite, being p and q positive integers relatively primes. In order to present our results we need some preliminary definitions and notations.

The unperturbed system (2) has a unique singular point, the origin with eigenvalues:

$$\pm\sqrt{a_\theta}i, \quad \pm\sqrt{a_\phi}i.$$

Consequently this system in the phase space $(x,dx/dt,y,dy/dt)$ has two planes filled of periodic solutions with the exception of the origin. These periodic solutions have periods

$$T_1=\frac{2\pi}{\sqrt{a_\theta}} \quad \text{or} \quad T_2=\frac{2\pi}{\sqrt{a_\phi}},$$

accordingly they belong to the plane associated to the eigenvectors with eigenvalues $\pm\sqrt{a_\theta}i$ or $\pm\sqrt{a_\phi}i$, respectively. We shall study which of these periodic solutions persist for the perturbed system (3) when the parameter ε is sufficiently small and the perturbed functions F_i for $i=1,2$ have period either pT_1/q, or pT_2/q, where p and q are positive integers relatively prime. On the other hand, in this work, the periods of the unperturbed system are not commensurable.

We define the functions:

$$\mathcal{F}_1(X_0,Y_0)=\frac{1}{2\pi p}\int_0^{pT_1}\sin(\sqrt{a_\theta}t)F_1(t,\Delta_1,\Delta_2,0,0)\,dt,$$
$$\mathcal{F}_2(X_0,Y_0)=\frac{\sqrt{a_\theta}}{2\pi p}\int_0^{pT_1}\cos(\sqrt{a_\theta}t)F_1(t,\Delta_1,\Delta_2,0,0)\,dt,$$
$$\mathcal{G}_1(Z_0,W_0)=\frac{1}{2\pi p}\int_0^{pT_2}\sin(\sqrt{a_\phi}t)F_2(t,0,0,\Delta_3,\Delta_4)\,dt,$$
$$\mathcal{G}_2(Z_0,W_0)=\frac{\sqrt{a_\phi}}{2\pi p}\int_0^{pT_2}\cos(\sqrt{a_\phi}t)F_2(t,0,0,\Delta_3,\Delta_4)\,dt, \qquad (4)$$

with

$$\Delta_1=X_0\cos(\sqrt{a_\theta}t)+\frac{Y_0}{\sqrt{a_\theta}}\sin(\sqrt{a_\theta}t),$$
$$\Delta_2=Y_0\cos(\sqrt{a_\theta}t)-\sqrt{a_\theta}X_0\sin(\sqrt{a_\theta}t),$$
$$\Delta_3=Z_0\cos(\sqrt{a_\phi}t)+\frac{W_0}{\sqrt{a_\phi}}\sin(\sqrt{a_\phi}t),$$
$$\Delta_4=W_0\cos(\sqrt{a_\phi}t)-\sqrt{a_\phi}Z_0\sin(\sqrt{a_\phi}t).$$

A zero (X_0^*,Y_0^*) of the non-linear system

$$\mathcal{F}_1(X_0,Y_0)=0, \quad \mathcal{F}_2(X_0,Y_0)=0, \qquad (5)$$

such that

$$\det\left(\frac{\partial(\mathcal{F}_1,\mathcal{F}_2)}{\partial(X_0,Y_0)}\bigg|_{(X_0,Y_0)=(X_0^*,Y_0^*)}\right)\neq 0$$

is called a *simple zero* of system (5). Similarly, a zero (Z_0^*,W_0^*) of the non-linear system

$$\mathcal{G}_1(Z_0,W_0)=0, \quad \mathcal{G}_2(Z_0,W_0)=0, \qquad (6)$$

such that

$$\det\left(\frac{\partial(\mathcal{G}_1,\mathcal{G}_2)}{\partial(Z_0,W_0)}\bigg|_{(Z_0,W_0)=(Z_0^*,W_0^*)}\right)\neq 0$$

is called a *simple zero* of system (6).

Our main result on the periodic solutions of the perturbed dumbbell satellite (3) is the following.

Theorem 1. *Assume that the functions F_1 and F_2 of the perturbed dumbbell satellite with equations of motion (3) are periodic in t of period pT_1/q with p and q positive integers relatively prime. Then for $\varepsilon\neq 0$ sufficiently small and for every simple zero $(X_0^*,Y_0^*)\neq(0,0)$ of the non-linear system (5), the perturbed dumbbell satellite (3) has a periodic solution $(x(t,\varepsilon),(dx/dt)(t,\varepsilon),y(t,\varepsilon),(dy/dt)(t,\varepsilon))$ tending when $\varepsilon\to 0$ to $(X_0^*,Y_0^*,0,0)$.*

Theorem 1 is proved in Section 2. Its proof is based in the averaging theory for computing periodic solutions, see Appendix of the paper.

We provide an application of Theorem 1 in the following corollary, which will be proved in Section 3.

Corollary 2. *The system*

$$\frac{d^2x}{dt^2}+a_\theta x=\varepsilon(1-xy^2-x^2)\left(\frac{dx}{dt}\right)^3,$$
$$\frac{d^2y}{dt^2}+a_\phi y=\varepsilon\left(x+x^2y+y\left(\frac{dx}{dt}\right)^2+\sin(\sqrt{a_\theta}t)\left(\frac{dx}{dt}\right)^2\right)$$

for $\varepsilon\neq 0$ sufficiently small has one periodic solution with

$$\lim_{\varepsilon\to 0}\left(x(0,\varepsilon),\frac{dx}{dt}(0,\varepsilon),y(0,\varepsilon),\frac{dy}{dt}(0,\varepsilon)\right)=\left(\frac{\sqrt{38+648\mu^2}}{3\sqrt{1+12\mu^2}},\frac{2\sqrt{6}}{3\sqrt[4]{1+12\mu^2}},0,0\right).$$

Similarly we obtain the following result.

Theorem 3. *Assume that the functions F_1 and F_2 of the perturbed dumbbell satellite with equations of motion (3) are periodic in t of period pT_2/q with p and q positive integers relatively prime. Then for $\varepsilon\neq 0$ sufficiently small and for every simple zero $(Z_0^*,W_0^*)\neq(0,0)$ of the non-linear system (6), the perturbed dumbbell satellite (3) has a periodic solution $(x(t,\varepsilon),(dx/dt)(t,\varepsilon),y(t,\varepsilon),(dy/dt)(t,\varepsilon))$ tending when $\varepsilon\to 0$ to $(0,0,Z_0^*,W_0^*)$.*

Theorem 3 is also proved in Section 2.

On the other hand, we provide an application of Theorem 3 in the next corollary, which will be proved in Section 3.

Corollary 4. *The system*

$$\frac{d^2x}{dt^2}+a_\theta x=\varepsilon\left(xy+\left(\frac{dx}{dt}\right)^3\frac{dy}{dt}+y\frac{dy}{dt}+\sin(\sqrt{a_\theta}t)\right),$$
$$\frac{d^2y}{dt^2}+a_\phi y=\varepsilon\frac{dy}{dt}\left(y^2-\frac{1}{a_\phi}-\frac{1}{4}+x^2\frac{dx}{dt}\frac{dy}{dt}\right)$$

for $\varepsilon\neq 0$ sufficiently small has two periodic solutions with

$$\lim_{\varepsilon\to 0}\left(x(0,\varepsilon),\frac{dx}{dt}(0,\varepsilon),y(0,\varepsilon),\frac{dy}{dt}(0,\varepsilon)\right)=(0,0,1,2)$$

and

$$\lim_{\varepsilon\to 0}\left(x(0,\varepsilon),\frac{dx}{dt}(0,\varepsilon),y(0,\varepsilon),\frac{dy}{dt}(0,\varepsilon)\right)=\left(0,0,\frac{2}{\sqrt{a_\phi}},\sqrt{a_\phi}\right).$$