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A reaction–diffusion equation and its traveling wave solutions

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ABSTRACT

In the present paper, we study a non-linear reaction–diffusion equation, which can be considered as a generalized Fisher equation. An exact solution and traveling wave solutions to the generalized Fisher equation are obtained by means of the Cole–Hopf transformation and the Lie symmetry method.

1. Introduction

The problems of the propagation of non-linear waves have fascinated scientists for over two hundred years [1,2]. Modern theories describe non-linear waves and coherent structures in a diverse variety of fields, including general relativity, high energy particle physics, plasmas, atmosphere and oceans, animal dispersal, random media, chemical reactions, biology, non-linear electrical circuits, and non-linear optics. Nowadays it has been universally acknowledged in the physical, chemical and biological communities that the reaction–diffusion equation plays an important role in dissipative dynamical systems. Typical examples are provided by the fact that there are many phenomena in biology where a key element or precursor of a developmental process seems to be the appearance of a traveling wave of chemical concentration (or mechanical deformation). When reaction kinetics and diffusion are coupled, traveling waves of chemical concentration can effect a biochemical change much faster than straight diffusional processes. This usually gives rise to reaction–diffusion equations which in one dimensional space takes the form

$$\frac{\partial u}{\partial t} = k_0 \frac{\partial^2 u}{\partial x^2} + F(u), \tag{1}$$

for a chemical concentration u, where k_0 is the diffusion coefficient, and $f(u)$ represents the kinetics.

When $F(u)$ is linear, i.e., $F(u)=k_2u+k_1$, where both k_1 and k_2 are constants, then in many instances Eq. (1) can be solved by the separation of variables methods. However if, as in many of the applications considered in [3], $F(u)$ is non-linear, then the problem is much more intractable. Indeed, it is not usually possible to obtain general exact analytical traveling wave solutions and one must analyze such problems numerically [4]. Despite this, however, under some particular circumstances, many non-linear evolutionary equations have traveling wave solutions of special types, which are of fundamental importance to our understanding of biological phenomena modeled evolutionary equations. The classic and simplest case of the non-linear reaction–diffusion equation is when $F(u)=k_3u(1-u)$, which is the so-called Fisher equation. It was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population [5]. (Although this equation is now referred to as the Fisher equation, the discovery, investigation and analysis of traveling waves in chemical reactions was first presented by Luther at a conference [6]. There, he stated that the wave speed is

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a simple consequence of the differential equations. This recently re-discovered paper has been translated into English by Arnold et al. [7] and Luther's remarkable discovery and analysis of chemical waves has been put in a modern context by Showalter and Tyson [8].) In the 20th century, the Fisher equation has became the basis for a variety of models for spatial spread. The typical examples are that Aoki discussed gene-culture waves of advance [9] and Ammerman and Cavali-Sforza, in an interesting direct application of the model, applied it to the spread of early farming in Europe [10,11]. Meanwhile, the qualitative analysis in the phase plane and traveling wave solutions of the Fisher equation have been widely investigated. The seminal and now classical references are that by Kolmogorov, Petrovsky and Piscunov [12], Albowitz and Zeppetella [13], Fife [14] and Britten [15]. In [12], Kolmogorov et al. showed that any initial concentration which is one for large negative spatial variable x and vanishes for large x, evolves to a traveling wavefront with minimal velocity $v=2\sqrt{k_0}$. Different initial values propagate with different traveling waves, depending on the behavior at $x\to\pm\infty$. The first explicit analytic form of a cline solution for the Fisher equation was obtained by Albowitz and Zeppetella making use of the Painlevé analysis [13]. A full discussion of this equation and an extensive bibliography can be seen in [14,15]. The singular property, auto-Bäcklund transformation and analytic solutions including some heteroclinic and homoclinic solutions of the Fisher equation were obtained by Guo and Chen via the expanded Painlevé for carrier flow equation in semiconductor devices [16,17]. A discrete singular convolution algorithm was introduced to solve Fisher's equation and predicted long-time traveling wave behavior by Zhao and Wei [18].

In the present work, we consider Eq. (1) with $F(u)=u(\mu+\beta u-\gamma u^2)$, namely

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}+u(\mu+\beta u-\gamma u^2), \tag{2}$$

where α, β, μ and γ are real constants. This equation can be regarded as a generalization of the Fisher equation, which is used as a density-dependent diffusion model, in the one-dimensional situation, for studying insect and animal dispersal with growth dynamics [3], and as a genetic model arising from the classical theory of population genetics and combustion [19,20]. During the past decade, considerable attention has been received to exact solutions and traveling wave solutions of some special cases of Eq. (2). When $\beta=0$, exact solutions were obtained by Clarkson and Mansfield [21] using the non-classical method. When both β and γ are non-zero, exact solutions of Eq. (2) have been found by Chen and Guo using a truncated Painlevé expansion [22], by Chowdhury [23] and Estévez and Gordoa [24] using a complete Painlevé test, and by Clarkson and Mansfield [21]. More profound results have been established by Clarkson and Mansfield making use of the non-classical reductions method [25]. Herrera et al. [26] obtained traveling wave solutions of (2) when $p=2$ in their equations. Kudryashov [27] derived an exact solitary wave solution to Eq. (2) by utilizing the Riccati equation and the Jacobi elliptic function. The study of the properties of the traveling waves and their applications were undertook by de Pablo and Sanchez [28]. Note that since two non-linearities occur in Eq. (2), in general it is not integrable. Therefore, to seek exact solutions of Eq. (2), qualitative analysis together with ingenious mathematical techniques for treating such non-linear system appears to be more powerful and important. Recently, qualitative results for some physical and biological systems have been studied extensively [29–32] and some innovative mathematical methods, such as the Lie group analysis and symmetry method have been developed and widely applied to many non-linear systems [33–35].

Our goal in this paper is to find exact solutions and traveling wave solutions to Eq. (2) under certain parametric conditions. The rest of the paper is organized as follows. In Section 2, we consider a special case of Eq. (2) where $\mu=0$. A traveling wave solution is found by utilizing the Cole–Hopf transformation, and an exact solution is presented by means of the Lie symmetry method. In Section 3, we focus on traveling wave solutions in terms of elliptic functions for Eq. (2). Section 4 is a brief conclusion.

2. Exact solution in the case $\mu=0$

In this section we mainly study exact solutions for Eq. (2) when $\mu=0$. Make a Cole–Hopf transformation:

$$u=R(\ln p)_x, \tag{3}$$

where R is a real constant, and p is a function of x and t to be determined. Substitution of (3) into Eq. (2) yields

$$3p_xp_{xx}p^2-p_xp_tp^2-p_{xxx}p^3-2(p_x)^3p+p_{xt}p^3-R\beta(p_x)^2p^2+R^2\gamma(p_x)^3p=0. \tag{4}$$

To reduce (4) to a bilinear equation as follows, we set $R=\pm\sqrt{2/\gamma}$

$$3p_xp_{xx}-p_xp_t-p_{xxx}p+p_{xt}p-R\beta(p_x)^2=0. \tag{5}$$

Assume that Eq. (5) admits the solution of the form

$$p(x,t)=k_1e^{(m_1x+n_1t)}+k_2\cdot e^{(m_2x+n_2t)}+k_3\cdot e^{(m_3x+n_3t)}, \tag{6}$$

where k_i ($i=1, 2, 3$), and m_j and n_j ($i=1, 2, 3$) are constants. Substituting (6) into (5) and equating the corresponding coefficients of the resulting exponential functions, we get an algebraic system

$$\begin{cases} 3m_im_j(m_i+m_j)-(m_i)^3-(m_j)^3-2R\beta m_im_j \\ \quad +(n_i-n_j)(m_i-m_j)=0 \quad (1\le i<j\le 3), \\ 2(m_q)^3-R\beta(m_q)^2=0 \quad (1\le q\le 3). \end{cases} \tag{7}$$

System (7) can be solved with the aid of Maple. Changing to our original variables, we obtain that Eq. (2) has exact solutions as

$$\begin{aligned} u_1(x,t)&=\frac{\beta}{\gamma}\cdot\frac{k_2\cdot e^{\sqrt{(\beta^2/2\alpha\gamma)}x+(\beta^2/2\gamma)t}}{k_2\cdot e^{\sqrt{(\beta^2/2\alpha\gamma)}x+(\beta^2/2\gamma)t}+(k_1+k_3)} \\ &=\frac{\beta}{\gamma}\cdot\frac{e^{\sqrt{(\beta^2/2\alpha\gamma)}x+(\beta^2/2\gamma)t}}{e^{\sqrt{(\beta^2/2\alpha\gamma)}x+(\beta^2/2\gamma)t}+C_1}, \end{aligned} \tag{8}$$

and

$$u_2(x,t)=\frac{\beta}{\gamma}\cdot\frac{e^{-\sqrt{(\beta^2/2\alpha\gamma)}x+(\beta^2/2\gamma)t}}{e^{-\sqrt{(\beta^2/2\alpha\gamma)}x+(\beta^2/2\gamma)t}+C_2}, \tag{9}$$

where C_1 and C_2 are arbitrary. Note that (8) and (9) are actually traveling wave solutions. When both C_1 and C_2 are positive, $u_1(x, t)$ and $u_2(x, t)$ each describe a kind-profile traveling wave (see Fig. 1). These two solutions are monotone with respect to $\xi=\pm\sqrt{(\beta^2/2\alpha\gamma)}x+(\beta^2/2\gamma)t$. They are analytic on the whole (x, t)-plane, but blow up at infinite points of (x, t) when both C_1 and C_2 are negative.

Now let us briefly state the general idea of the Lie symmetry method for partial differential equations (PDEs) [33–35]. Here we only consider PDEs with one dependent variable u and two independent variables x and t. A point transformation is a diffeomorphism

$$T:(x,t,u)\mapsto(\tilde{x}(x,t,u),\tilde{t}(x,t,u),\tilde{u}(x,t,u)),$$

which maps the surface $u=u(x, t)$ to the surface

$$\tilde{x}=\tilde{x}(x,t,u(x,t)),\quad \tilde{t}=\tilde{t}(x,t,u(x,t)),\quad \tilde{u}=\tilde{u}(x,t,u(x,t)). \tag{10}$$