International Journal of Non-Linear Mechanics 49 (2013) 159–169

Contents lists available at SciVerse ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

# Solution of the model of beam-type micro- and nano-scale electrostatic actuators by a new modified Adomian decomposition method for nonlinear boundary value problems

Jun-Sheng Duan [a,b,*], Randolph Rach [c], Abdul-Majid Wazwaz [d]

[a] College of Science, Shanghai Institute of Technology, Fengxian District, Shanghai 201418, PR China
[b] Institute of Mathematical Sciences, Huaqiao University, Quanzhou, Fujian 362021, PR China
[c] 316 South Maple Street, Hartford, MI 49057-1225, USA
[d] Department of Mathematics, Saint Xavier University, 3700 West 103rd Street, Chicago, IL 60655-3105, USA

ARTICLE INFO

*Article history:*
Received 19 April 2012
Received in revised form
16 September 2012
Accepted 7 October 2012
Available online 23 October 2012

*Keywords:*
Adomian decomposition method
Nonlinear boundary value problems
Casimir force
Van der Waals force
Micro-electromechanical systems (MEMS)
Nano-electromechanical systems (NEMS)

ABSTRACT

In this paper we solve the common nonlinear boundary value problems (BVPs) of cantilever-type micro-electromechanical system (MEMS) and nano-electromechanical system (NEMS) using the distributed parameter model by the Duan–Rach modified Adomian decomposition method (ADM). The nonlinear BVPs that are investigated include the cases of the single and double cantilever-type geometries under the influence of the intermolecular van der Waals force and the quantum Casimir force for appropriate distances of separation. The new Duan–Rach modified ADM transforms the nonlinear BVP consisting of a nonlinear differential equation subject to appropriate boundary conditions into an equivalent nonlinear Fredholm–Volterra integral equation before designing an efficient recursion scheme to compute approximate analytic solutions without resort to any undetermined coefficients. The new approach facilitates parametric analyses for such designs and the pull-in parameters can be estimated by combining with the Padé approximant. We also consider the accuracy and the rate of convergence for the solution approximants of the resulting Adomian decomposition series, which demonstrates an approximate exponential rate of convergence. Furthermore we show how to easily achieve an accelerated rate of convergence in the sequence of the Adomian approximate solutions by applying Duan's parametrized recursion scheme in computing the solution components. Finally we compare the Duan–Rach modified recursion scheme in the ADM with the method of undetermined coefficients in the ADM for solution of nonlinear BVPs to illustrate the advantages of our new approach over prior art.



## 1. Introduction

Beam-type electrostatic actuators have become one of the common components in constructing micro-electromechanical system (MEMS) and nano-electromechanical system (NEMS) [1]. A beam-type actuator is constructed from a conductive electrode suspended over a conductive substrate. A voltage difference between the two electrodes causes the upper movable electrode to deflect towards the ground electrode. At a critical voltage the movable electrode becomes unstable and pulls-in onto the substrate. The parameters of the actuator in this state, such as the voltage and deflection, are called the pull-in parameters.

The pull-in behavior of MEMS actuators has been previously studied in [2,3]. The intermolecular and quantum forces were neglected in these studies of micromechanical actuators on the basis of physical scale.

With recent developments in nanotechnology, many researchers have focused on investigation of the effect of intermolecular forces on the performance of electromechanical actuators. For separations below 20 nm the force between two surfaces is known as the intermolecular van der Waals attraction, which varies as the inverse cube of the separation [4–6]. When the separation is above 20 nm, the force between two surfaces can be described by the quantum Casimir effect, which is proportional to the inverse fourth power of the separation [6–11].

Beam-type NEMS actuators are modeled by a beam of length $L$ with a uniform rectangular cross section of width $w$ and thickness $h$, which is suspended over a conductive substrate and separated by a dielectric spacer. They are classified as the cantilever NEMS and the double cantilever NEMS according to the design of the dielectric support structure; see figures in [2,9].

The governing equation for the distributed parameter model, based on the Euler–Bernoulli beam assumptions, may be

* Corresponding author. at: College of Science, Shanghai Institute of Technology, Fengxian District, Shanghai 201418, PR China.
*E-mail addresses:* duanjssdu@sina.com (J.-S. Duan), tapstrike@triton.net (R. Rach), wazwaz@sxu.edu (A.-M. Wazwaz).

0020-7462/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijnonlinmec.2012.10.003

written as [6,9]

$$\tilde{E}I\frac{d^4Y}{dX^4}=F_{elec}+F_K, \tag{1.1}$$

where $Y$ is the deflection of the beam, $X$ is the position along the beam as measured from the clamped end, $I$ is the moment of inertia of the beam cross section, and $\tilde{E}$ is the effective modulus. The effective modulus $\tilde{E}$ simply becomes Young's modulus $E$ for narrow beams ($w<5h$) or the plate modulus $E/(1\text{-}\nu^2)$, where $\nu$ is the Poisson ratio, for wide beams ($w\geq 5h$). On the right side of Eq. (1.1), $F_{elec}$ and $F_K$ ($K=3,4$) are the electrostatic and intermolecular/quantum forces per unit length of the beam, respectively. The van der Waals force ($F_3$) and the Casimir force ($F_4$) are taken into account considering their range of application. For this brief investigation, we omit consideration of the transition regime.

The first-order fringing field correction of the electrostatic force per unit length of the beam is

$$F_{elec}=\frac{\varepsilon_0 wV^2}{2(s\text{-}Y)^2}\left(1+\frac{0.65(s\text{-}Y)}{w}\right), \tag{1.2}$$

where $\varepsilon_0=8.854\times 10^{-12}\ \mathrm{C^2\ N^{-1}\ m^{-2}}$ is the permittivity of vacuum, $V$ is the applied voltage and $s$ is the original gap between the two electrodes if there were no deflection.

The van der Waals force per unit length of the beam [4–6,12] is

$$F_3=\frac{Aw}{6\pi(s\text{-}Y)^3}, \tag{1.3}$$

where $A$ is the Hamaker constant.

The Casimir force per unit length of the beam [8,9,13] is

$$F_4=\frac{\pi^2\hbar\nu w}{240(s\text{-}Y)^4}, \tag{1.4}$$

where $\hbar=1.055\times 10^{-34}\ \mathrm{J\,s}$ is the reduced Planck's constant, a measure of quantum action, and $\nu=2.998\times 10^{8}\ \mathrm{m\,s^{-1}}$ is the speed of light.

Substituting Eqs. (1.2), (1.3) and (1.4) into Eq. (1.1), and introducing the dimensionless variables $y=Y/s$ and $x=X/L$, we transform the governing equation to the dimensionless form

$$\frac{d^4y}{dx^4}=\frac{\alpha_K}{(1\text{-}y(x))^K}+\frac{\beta}{(1\text{-}y(x))^2}+\frac{\gamma}{1\text{-}y(x)}, \tag{1.5}$$

where the index $K$ equals 3 for the van der Waals force and 4 for the Casimir force. The dimensionless parameters appearing in Eq. (1.5) are

$$\alpha_3=\frac{L^4Aw}{6\pi\tilde{E}Is^4},\quad \alpha_4=\frac{\pi^2L^4\hbar\nu w}{240\tilde{E}Is^5},\quad \beta=\frac{L^4\varepsilon_0 wV^2}{2\tilde{E}Is^3},\quad \gamma=\eta\beta=\frac{0.65s}{w}\beta.$$

The boundary conditions are

$$y(0)=0,\quad y'(0)=0,\quad y''(1)=0,\quad y'''(1)=0, \tag{1.6}$$

for a cantilever NEMS and

$$y(0)=0,\quad y'(0)=0,\quad y(1)=0,\quad y'(1)=0, \tag{1.7}$$

for a double cantilever NEMS.

Let $u(x)=1\text{-}y(x)$, then $u(x)$ is the dimensionless gap between the movable electrode and the substrate. We then rewrite Eq. (1.5) as

$$\frac{d^4u}{dx^4}=-\frac{\alpha_K}{u(x)^K}-\frac{\beta}{u(x)^2}-\frac{\gamma}{u(x)}. \tag{1.8}$$

We remark that this trivial transformation is unnecessary in the ADM, and have adopted it for convenience of comparison with previously published solutions [9,14].

The boundary conditions are then

$$u(0)=1,\quad u'(0)=0,\quad u''(1)=0,\quad u'''(1)=0, \tag{1.9}$$

for a cantilever NEMS and

$$u(0)=1,\quad u'(0)=0,\quad u(1)=1,\quad u'(1)=0, \tag{1.10}$$

for a double cantilever NEMS.

We remark that when $\alpha_K=0$, Eq. (1.8) corresponds to the case of the MEMS, i.e. the intermolecular and quantum forces are neglected.

The boundary value problems (BVPs) for the MEMS and NEMS have been previously solved by using the method of undetermined coefficients to determine the constants of integration in the Adomian decomposition method (ADM) [2,5,9,12,14–16]. The aim of this paper is to solve the above model of the nonlinear electrostatic NEMS by using the new Duan–Rach modification of the ADM for solution of nonlinear BVPs [17].

The ADM is a practical technique for solving functional equations. The method, which requires neither linearization nor perturbation, efficiently works for a wide class of initial value problems (IVPs) or BVPs, encompassing linear, nonlinear, and even stochastic systems [18–33]. The method also does not resort to the Green function concept, which greatly facilitates analytic approximations and numerical computations [34–36].

The goal of the ADM is to provide a systematic approach for approximate analytic solutions of nonlinear and stochastic operator equations, including differential, integral and integro-differential equations. Furthermore we note that the ADM provides a convenient approach in computing the error analysis and convergence analysis in the absence of a priori known exact solutions, which is most often the case for nonlinear engineering design problems, e.g. the error remainder function and the maximal error remainder parameter, respectively; see Appendix A. For a comprehensive bibliography and recent developments of the ADM see [37,38].

The ADM decomposes the solution into a series

$$u(x)=\sum_{m=0}^{\infty}u_m(x),$$

where the components are determined by an appropriate recursion scheme. The analytic nonlinearity $\mathcal{N}u(x)$ is decomposed into a series of Adomian polynomials that are tailored to the specific nonlinearity

$$\mathcal{N}u(x)=\sum_{m=0}^{\infty}A_m(x),$$

where the Adomian polynomials $A_m(x)$ only depend on the solution components $u_0(x),u_1(x),\ldots,u_m(x)$, for which the definitional formula was first published by Adomian and Rach in 1983 [18]:

$$A_m(x)=\frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}\mathcal{N}\left(\sum_{k=0}^{\infty}\lambda^k u_k(x)\right)\Bigg|_{\lambda=0}. \tag{1.11}$$

For convenient reference, we list the first five Adomian polynomials for the simple nonlinearity $\mathcal{N}u(x)=f(u(x))$,

$$A_0=f(u_0),$$
$$A_1=f'(u_0)u_1,$$
$$A_2=f'(u_0)u_2+f''(u_0)\frac{u_1^2}{2!},$$
$$A_3=f'(u_0)u_3+f''(u_0)u_1u_2+f'''(u_0)\frac{u_1^3}{3!},$$
$$A_4=f'(u_0)u_4+f''(u_0)\left(\frac{u_2^2}{2!}+u_1u_3\right)+f'''(u_0)\frac{u_1^2u_2}{2!}+f^{(4)}(u_0)\frac{u_1^4}{4!}.$$

Several algorithms [18,21,23,25,28,39–48] for symbolic programming have since been devised to efficiently generate the Adomian polynomials quickly and to high orders. For example, a convenient formula for the Adomian polynomials is Rach's Rule, which reads [21, p. 16; 23, p. 51])