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Stationary response of Duffing oscillator with hardening stiffness and fractional derivative

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ABSTRACT

The stationary response of Duffing oscillator with hardening stiffness and fractional derivative under Gaussian white noise excitation is studied. First, the term associated with fractional derivative is separated into the equivalent quasi-linear dissipative force and quasi-linear restoring force by using the generalized harmonic balance technique, and the original system is replaced by an equivalent nonlinear stochastic system without fractional derivative. Then, the stochastic averaging method of energy envelope is applied to the equivalent nonlinear stochastic system to yield the averaged Itô equation of energy envelope, from which the corresponding Fokker–Planck–Kolmogorov (FPK) equation is established and solved to obtain the stationary probability densities of the energy envelope and the amplitude envelope. The accuracy of the analytical results is validated by those from the Monte Carlo simulation of original system.

1. Introduction

In the last several decades, fractional derivative-based technique has been generally recognized as the well-established tool to model the constitutive behavior of viscoelastic materials. In this regard, original contribution was owed to Gemant [1], who suggested a fractional derivative constitutive relationship to model cyclic-deformation tests performed on viscoelastic material specimens. Later, Caputo [2] reported experimental validity when they used fractional derivatives for the description of the behavior of viscoelastic materials. Moreover, Bagley and Torvik [3] have provided the theoretical basis for the use of the fractional derivative models to characterize viscoelasticity in the early 1980s. So far, many researchers such as Koh and Kelly [4], Markris and Constantious [5], Pritz [6], Friedrich et al. [7], Mainardi [8], Papoulia and Kelly [9], Rossikhin and Shitikova [10] have given further insight into the potential of fractional derivative when applied to the viscoelasticity modeling. In particular, Gorenflo and Mainardi [11], Kempfle et al. [12], and Rossikhin and Shitikova [13,14] have provided an excellent review of the research in this field.

Parallelly, many authors have put forward the analysis solutions of deterministic dynamic systems involving elements described by fractional derivatives. Among them, Padovan and Sawicki [15] discussed the long time behavior of Duffing oscillator endowed with fractional derivative damping using perturbation method and examined the influence of fractional order on the frequency amplitude response behavior. Pa1falvi [16] provided a computationally efficient solution method for the fractionally damped vibration equation using the Adomian decomposition method and Taylor series. Leung and Guo [17] proposed an improved harmonic balance method for autonomous and non-autonomous systems with fractional derivative damping and examined the interaction among the excitation frequency, fractional order, amplitude, phase angle and the frequency amplitude response. Shen [18] studied the primary resonance of Duffing oscillator with fractional derivative using standard averaging method. They pointed out that the fractional derivative term could both affect the viscous damping and the linear stiffness. Kovacic and Zukovic [19] investigated the free oscillators with a power-form restoring and fractional derivative damping using the averaging method, with particular attention to the effects of fractional derivative order on the amplitude and frequency of oscillations.

Actually, stochastic perturbations are ubiquitous. So it is necessary to compute the response to stochastic excitations. To this aim, a frequency domain technique has been pursued by Spanos and Zeldin [20] and Rüdinger [21]. Alternatively, a time

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domain Duhamel integral closed-form expression has been obtained to analyze the stochastic response by using the Laplace transform [22,23] technique or the Fourier transform [24] technique. Recently, a stochastic averaging method based on the generalized harmonic function has been explored to investigate the stochastic dynamics including response [25,26], first passage failure [27], and stochastic stability [28] of strongly nonlinear oscillators with fractional derivative damping. Furthermore, Ref. [29] obtained a general frequency domain solution based on statistical linearization; results were presented for a Duffing oscillator with fractional derivative damping subjected to external Gaussian white noise excitation [29]. Ref. [30] computed the response a SDOF linear system with fractional derivative damping subjected to stationary and non-stationary random excitations, in which the key idea was generalized to the fractionally damped Duffing oscillator subjected to a stochastic input [31].

In the preceding works on stochastic dynamics, however, the term associated with the fractional derivative was simply considered as the special damping force. Note that the so-called fractional derivative damping not only serves as the role of classical damping force but also contributes to the elastic restoring force [18,32]. Thus, simply considering the fractional derivative term as the dissipation force is insufficient or even incorrect. In this paper, the stationary response of Duffing oscillator with hardening stiffness and fractional derivative subjected to Gaussian white noise excitation is investigated. The key idea is to decouple the fractional derivative term into the equivalent quasi-linear dissipative force and quasi-linear restoring force based on the generalized harmonic balance technique and to apply the stochastic averaging method of energy envelope to the equivalent nonlinear stochastic system without fractional derivative. Some parameters including the order of fractional derivative, the magnitude of non-linearity, the coefficient of fractional derivative term and the intensity of excitation are examined. The analytical results are compared to the Monte Carlo simulation data and the old analytical results [25].

2. Equivalent nonlinear stochastic system

Consider a Duffing oscillator with hardening stiffness and fractional derivative under stochastic excitation as shown in Fig. 1. The motion of the system is governed by

$$m\ddot{X}(t)+c\dot{X}(t)+kX(t)+\chi D^{\alpha}X(t)=\xi(t) \tag{1}$$

where $X(t)$ is the displacement; m, c, $k=k_0+k_1x^2(t)$ are mass, damping and hardening spring, respectively; m, c, k_0 and k_1 are all positive parameters. $D^{a}X(t)$ denotes the fractional derivative operation of Riemann–Liouville's definition, i.e.,

$$D^{\alpha}X(t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{X(t-\tau)}{\tau^{\alpha}}\mathrm{d}\tau;\quad 0<\alpha\le 1 \tag{2}$$

$\xi(t)$ is a stationary Gaussian white noise with correlation function $E[\xi(t)\xi(t+\tau)]=2D_1\delta(\tau)$.

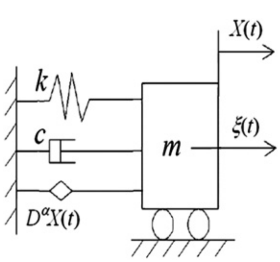

Fig. 1. The Duffing oscillator with hardening stiffness and fractional derivative under random excitation.

Using the following transformation of coordinates:

$$\omega_0=\sqrt{k_0/m};\quad 2\varepsilon\zeta=(c/m);\quad \alpha_0=(k_1/m);\quad \varepsilon\chi_1=(\chi/m);$$
$$\sqrt{\varepsilon f}=1/m \tag{3}$$

where ε is a small positive parameter. Eq. (1) can be rewritten as

$$\ddot{X}(t)+2\varepsilon\zeta\dot{X}(t)+\omega_0^2X+\alpha_0X^3+\varepsilon\chi_1D^{\alpha}X(t)=\sqrt{\varepsilon f}\,\xi(t) \tag{4}$$

The solution can be assumed of the following form of the generalized harmonic function [33]:

$$X(t)=A(t)\cos\Theta(t)$$
$$\dot{X}(t)=-A(t)\nu(A,\Theta)\sin\Theta(t) \tag{5}$$

where

$$\Theta(t)=\Phi(t)+\Gamma(t)$$
$$\nu(A,\Theta)=\frac{\mathrm{d}\Phi}{\mathrm{d}t}=[(\omega_0^2+3\alpha_0A^2/4)(1+\eta\cos 2\Theta)]^{1/2}$$
$$\eta=\alpha_0A^2/(4\omega_0^2+3\alpha_0A^2)\le 1/3 \tag{6}$$

$\cos\Theta(t)$ and $\sin\Theta(t)$ are the so-called generalized harmonic functions, $\nu(A,\Theta)$ is instantaneous frequency of the oscillator and can be expanded into Fourier series, i.e.,

$$\nu(A,\Theta)=\sum\nolimits_{n=0}^{\infty}b_{2n}(A)\cos 2n\Theta \tag{7}$$

where

$$b_{2n}(A)=\frac{1}{2\pi}\int_0^{2\pi}\nu(A,\Theta)\cos 2\,n\Theta\mathrm{d}\Theta \tag{8}$$

Averaging $\nu(A,\Theta)$ with respect to Θ from 0 to 2π yeilds to the averaged frequency $\omega(A)$ as follows:

$$\omega(A)=b_0(A)=(\omega_0^2+3\alpha_0(A^2/4))^{1/2}(1-(\eta^2/16)) \tag{9}$$

$\Theta(t)$ in Eq. (6) can be approximated as

$$\Theta(t)\approx\omega(A)t+\Gamma(t) \tag{10}$$

As shown in Refs. [18,32], the term associated with fractional derivative contributes both to stiffness and damping. Based on the generalized harmonic function technique, such term can be replaced by the following forces containing a quasi-linear elastic force and a quasi-linear damping force:

$$\varepsilon\chi_1D^{\alpha}X(t)=\varepsilon C(A)\dot{X}(t)+\varepsilon K(A)X(t) \tag{11}$$

where

$$C(A)=-\frac{\chi_1}{\pi A\omega(A)}\int_0^{2\pi}D^{\alpha}(A\cos\Theta)\sin\Theta\mathrm{d}\Theta$$
$$=\chi_1\omega^{\alpha-2}(A)\left(b_0+\frac{b_2}{2}\right)\sin\left(\frac{\alpha\pi}{2}\right)$$

$$K(A)=\frac{\chi_1}{\pi A}\int_0^{2\pi}D^{\alpha}(A\cos\Theta)\cos\Theta\mathrm{d}\Theta$$
$$=\chi_1\omega^{\alpha-1}(A)\left(b_0-\frac{b_2}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right) \tag{12}$$

in which

$$\begin{aligned}b_0&=\omega(A)=(\omega_0^2+3\alpha_0(A^2/4))^{1/2}(1-(\eta^2/16))\\ b_2&=(\omega_0^2+3\alpha_0(A^2/4))^{1/2}((\eta/2+3)(\eta^3/64))\end{aligned} \tag{13}$$

Note that the detailed procedure of derivation is presented in Appendix.

The equivalent nonlinear stochastic system associated with system (4) is given by

$$\ddot{X}(t)+\varepsilon[2\zeta+C(A)]\dot{X}(t)+[(\varepsilon K(A)+\omega_0^2)X(t)+\alpha_0X^3(t)]=\sqrt{\varepsilon f}\,\xi(t) \tag{14}$$