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Flow of power-law fluid over a stretching surface: A Lie group analysis

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ABSTRACT

This paper investigates the boundary layer flow of power-law fluid over a permeable stretching surface. The use of Lie group analysis reveals all possible similarity transformations of the problem. The application of infinitesimal generator on the generalized surface stretching conditions leads to two possible surface conditions which leads to the possibility of two types of stretching velocities namely; the power-law and exponential stretching. The power-law stretching has already been discussed in the literature, however exponential stretching is investigated here for the first time. Interestingly, an exact analytical solution of the non-linear similarity equation for exponential stretching is developed for shear thinning fluid with power-law index $n=1/2$. This solution is further extended to a larger class of shear thinning fluids ($n \approx 1/2$) using perturbation method. In addition, the numerical solution for shear thinning fluid is also presented. The two solutions match excellently for shear thinning fluids. Analytical solution for shear thickening fluid is not tractable and the numerical solution is presented for completeness.

1. Introduction

The study of flow field due to stretching surface has found many applications in different fields of engineering and industry. The stretching of the plate is known to have a definite impact on the quality of the finished product. A number of real processes are thus undertaken using different stretching velocities such as linear, power-law and exponential. Specifically, such flows are generated in extrusion of polymers, fibers spinning, hot rolling, manufacturing of plastic and rubber sheet, continuous casting and glass blowing. For example, when a sheet of polymer is extruded continuously from a die, a boundary layer develops that grows along the sheet in the direction of its motion. A great deal of research in fluid mechanics is rightfully produced to model these problems and to provide analytical and numerical results for better understanding of the fluid behavior and adequate explanation of the experiments.

The history of stretching flows goes back to the celebrated papers by Sakiadis [1,2] who initiated the study of boundary layer behavior for the sheet moving with a constant velocity in a viscous fluid. The analytical solution for steady stretching of the surface was given by Crane [3]. Fox et al. [4] considered the effects of suction and injection on flow due to continuous moving surface. However, these studies were undertaken for linear stretching velocities. Gupta and Gupta [5] identified that stretching of the sheet may not necessarily be linear in real situations. The power-law stretching velocity was thus undertaken by Banks [6] and Ali [7]. Later on, exponential stretching for the viscous fluid was considered by Magyari and Keller [8] while Elbashbeshy [9] added the effects of wall suction in [8]. The underlying fluid in the preceding discussion was invariably viscous.

Many fluids such as blood, dyes, yoghurt, ketchup, shampoo, paint, mud, clay coatings, polymer melts, certain oils and greases etc. exhibit complex relations between stress and strain. Such fluids do not obey the Newton's law of viscosity and are usually called as non-Newtonian fluids. The flows of such fluids occur in a wide range of practical problems having vital importance in polymer depolarization, bubble columns, fermentation, composite processing, boiling, plastic foam processing, bubble absorption and many others. Many authors have investigated the phenomenon of stretching sheets in a viscoelastic fluid, we refer to [10,11] as well.

Although a second grade fluid model may predict the normal stress effects, it is not capable of describing shear thinning and shear thickening phenomena which are best described by power-law fluid. Pakdemirli [12,13] obtained similarity transformations and solutions of non-Newtonian power-law fluid for various geometries. Hassanien [14] and Zheng et al. [15] presented the

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numerical solution for continuously moving surface in a power-law fluid. Andersson and Kumaran [16] extended the work of Crane [3] to shear thinning and thickening fluid considering $n \approx 1$ and obtained the analytical solution for the case of linear stretching using perturbation method.

The exponential stretching of the sheets has significant importance in industry and engineering. It is well known that topological chaos depends upon the periodic motion of obstacles in a two-dimensional flow to form non-trivial braids. This motion is responsible for exponential stretching of material lines causing efficient mixing. Friedlander and Vishik [17] showed the existence of exponential stretching (the positivity of the Lyapunov exponent), as a necessary condition, for smooth flow in "fast" dynamo problem.

Motivated by these considerations, we present analytical and numerical solutions for exponentially stretching surface in a power-law fluid. There are a number of reasons that make this work significant. (a) To realize the importance of Lie group analysis [18–25] in working out all possible similarities for arbitrary stretching velocity in a power-law fluid. This leads us to discover the similarity transformations for power-law stretching and exponential stretching. The power-law stretching for power-law fluid has already been available in the literature while the exponential stretching is worked out for the first time; (b) having found the similarity transformations, the governing equations are transformed to self similar ordinary differential equation. The exact analytical solution (which is very rare) is now obtained for exponentially stretching of the surface for shear thinning fluid (power-law index $n=1/2$); (c) this solution is further extended to cover a larger class of shear thinning fluids i.e., $n \approx 1/2$, using perturbation method; (d) numerical solutions for shear thinning and thickening fluids are presented to establish the accuracy of the analytical solution. Analytical solution for shear thickening is not tractable and numerical solution is presented for the completeness.

2. Formulation of the problem

Let us consider a two dimensional laminar flow of a steady incompressible non-Newtonian power-law fluid over a permeable surface. The origin of the stationary Cartesian coordinate system is located at the leading edge of the surface undergoing a generalized stretching with the velocity $u_w(\tilde{x})$. The x-axis is along the surface and y-axis is taken normal to the surface. The suction or injection velocity through the surface is denoted by $v_w(\tilde{x})$. Using Boussinesq approximations the appropriate governing equations of continuity and momentum for power-law fluid are

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0, \tag{1}$$

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = \frac{1}{\rho}\frac{\partial \tau_{\tilde{x}\tilde{y}}}{\partial \tilde{y}}, \tag{2}$$

where $\tilde{u}$ and $\tilde{v}$ are the components of velocity in $\tilde{x}$ and $\tilde{y}$ directions, ρ is the fluid density and τ is the stress tensor. Following the Ostwald-de-Waele model equation with parameters defined by Bird et al. [26], the shear stress component of the stress tensor for power-law fluid can be written as

$$\tau_{\tilde{x}\tilde{y}} = -K\frac{\partial}{\partial \tilde{y}}\left(-\frac{\partial \tilde{u}}{\partial \tilde{y}}\right)^n,$$

where K is the consistency coefficient and n is the power-law index. In the above constitutive equation $n=1$ corresponds to Newtonian fluid, whereas $n<1$ and $n>1$ correspond to shear thinning and shear thickening fluids, respectively.

Substituting the value of stress, the governing momentum Eq. (2) becomes

$$\tilde{u}\frac{\partial \tilde{u}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{u}}{\partial \tilde{y}} = -\frac{K}{\rho}\frac{\partial}{\partial \tilde{y}}\left(-\frac{\partial \tilde{u}}{\partial \tilde{y}}\right)^n. \tag{3}$$

The boundary conditions of the problem are given by

$$\tilde{u} = U_0 u_w\left(\frac{\tilde{x}}{L}\right), \tilde{v} = V_0 v_w\left(\frac{\tilde{x}}{L}\right), \text{ at } \tilde{y}=0,$$
$$\tilde{u} \to 0, \text{ as } \tilde{y} \to \infty. \tag{4}$$

where U_0 and V_0 are the reference velocities and L is the characteristic length.

Introducing the following non-dimensional parameters (see [27]):

$$x = \frac{\tilde{x}}{L}, \quad = \frac{\tilde{y}}{L}\left(\frac{\rho U_0^{2-n} L^n}{K}\right)^{1/n+1},$$
$$u = \frac{\tilde{u}}{U_0}, \quad v = \frac{\tilde{v}}{U_0}\left(\frac{\rho U_0^{2-n} L^n}{K}\right)^{1/n+1}. \tag{5}$$

Eqs. (1), (3) and (4) take the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^n, \tag{7}$$

$$y=0; \quad u = u_w(x), \quad v = \frac{V_0 v_w(x)}{U_0}\left(\frac{\rho U_0^{2-n} L^n}{K}\right)^{1/n+1},$$
$$y \to \infty; \quad u = 0. \tag{8}$$

3. Symmetries of the problem

Using Lie group method [16–18] in Eqs. (6) and (7), the infinitesimal generator can be written as

$$X = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \varphi_1\frac{\partial}{\partial u} + \varphi_2\frac{\partial}{\partial v}.$$

We require that the Eqs. (6) and (7) remain invariant under the infinitesimal Lie point transformations given by

$$x^* = x + \varepsilon\xi_1(x,y,u,v) + O(\varepsilon^2),$$
$$y^* = y + \varepsilon\xi_2(x,y,u,v) + O(\varepsilon^2),$$
$$u^* = u + \varepsilon\varphi_1(x,y,u,v) + O(\varepsilon^2),$$
$$v^* = v + \varepsilon\varphi_2(x,y,u,v) + O(\varepsilon^2). \tag{9}$$

Employing a lengthy but straightforward algebra, the form of the infinitesimals is found to be

$$\xi_1 = a + bx, \quad \xi_2 = \frac{b+(n-2)c}{(n+1)}y + \gamma(x),$$
$$\varphi_1 = cu, \quad \varphi_2 = \frac{(2n-1)c - nb}{(n+1)}v + u\gamma'(x). \tag{10}$$

Eq. (10) reveals that there are three finite-parameters a, b and c and one infinite Lie group transformations ($\gamma(x)$) for this problem. The parameter 'a' corresponds to the translation in the variable x whereas the parameter 'b' corresponds to the scaling in the variables x, y and v and the parameter 'c' corresponds to the scaling in the variables y, u and v. We will discuss only those symmetries which leave the boundary conditions invariant. For