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Primary resonance of Duffing oscillator with two kinds of fractional-order derivatives

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ABSTRACT

In this paper, the primary resonance of Duffing oscillator with two kinds of fractional-order derivatives is investigated analytically. Based on the averaging method, the approximately analytical solution and the amplitude–frequency equation are obtained. The effects of the two kinds of fractional-order derivatives on the system dynamics are analyzed, and it is found that these two kinds of fractional-order derivatives could affect not only the linear viscous damping, but also the linear stiffness, which could be characterized by the equivalent damping coefficient and the equivalent stiffness coefficient. The different effects are analyzed based on these two deduced equivalent parameters, when the two fractional orders are limited in the typical intervals, i.e. $p_1 \in [0\ \ 1]$ and $p_2 \in [1\ \ 2]$. Moreover, the comparisons of the amplitude–frequency curves obtained by the approximately analytical solution and the numerical integration are fulfilled, and the results certify the correctness and satisfactory precision of the approximately analytical solution. Especially, the effects of the parameters in the second kind of fractional-order derivative are studied when the coefficient of the first kind of fractional-order derivative is zero or not. At last, two special cases for the coefficient of the second kind of fractional-order derivative are analyzed, which could make engineers obtain satisfactory vibration control performance and keep the frequency characteristic almost unchanged. These results are very useful in vibration control engineering.

1. Introduction

Fractional-order derivative and integral is a natural generalization of traditional integer-order counterpart, which was firstly presented in the late 1700s. Since then, a lot of investigations, both on general theory and engineering application of fractional-order derivative, has been carried out by many authors in different fields [1–34]. In the theoretical aspect, the results were focused on the definitions, properties, and efficient computation methods of the fractional-order derivative and integral. In the engineering application, two aspects were important and may be the most interesting subjects, i.e. the description of the memory and hereditary properties in various materials and processes, and the artificial introduction of the fractional-order feedback into the control engineering. In the engineering fields with fractional-order derivatives, the effects of the parameters in the fractional-order derivative on dynamical system were interesting and meaningful, and many issued works had been fulfilled on this subject.

Works on system dynamics with fractional-order derivatives may be divided into several groups, such as qualitative analysis, numerical computation, and analytical research on the approximate solution, etc. The qualitative analysis is primarily focused on the number and stability of equilibrium points and periodic solutions in the systems. For example, Machado and Galhano [10] analyzed statistical dynamics of a large number of micromechanical masses with backlash and impact, and found the coexistence of both integer and fractional properties in the global dynamics. Li et al. [11] studied the stable parameters range of the simplified Mathieu-type equation with fractional-order derivative, which originated from a simple supported viscoelastic column subjected to periodic axial force. By using the idea of stability switch, Wang and Hu [12] found that the fractional-order derivative in linear single degree-of-freedom (SDOF) dynamical system always acted as a damping force, and the unique equilibrium point would be asymptotically stable. Wang and Du [13] proved that the solution of a linear SDOF fractional-order oscillator without external excitation consisted of two parts, where the first one was similar to the case without fractional-order derivative, and the second one was a definite integral. Through the theorems on the stability of incommensurate fractional order systems, Tavazoei et al. [14] determined the parameters range where a van der Pol oscillator with a specific fractional order could perform as an undamped

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oscillator. Pinto and Machado [15,16] studied a van der Pol oscillator with complex-order and found multiple limit cycles coexisting in the system, and they analyzed the amplitudes and frequencies of the periodic solutions with and without external force.

Due to the complexity of fractional-order derivatives, numerical computation on the complicated non-linear dynamics phenomena such as bifurcation, chaos and synchronization was another important group in the dynamical system with fractional-order derivatives. Deng [18] designed a revised numerical scheme combining with the predictor-corrector approach, and presented the numerical error limit associated with the corresponding stability condition. Atanackovic and Stankovic [19] proposed a modified numerical procedure to solve fractional-order differential equations, and the test results on several examples verified the efficiency of the method. Cao et al. [20] simulated the fractional-order Duffing equation and investigated the effects of the fractional-order parameters on system dynamics using phase curves, bifurcation diagram and Poincaré map. Palfalvi [21] presented an improved Adomian decomposition method to solve fractional-order differential equation with sine excitation. Sheu et al. [22] solved the fractional-order damped Duffing equations by transforming them into a set of fractional-order integral equations. Wu et al. [23], Chen and Chen [24], and Lu [25] studied the synchronization phenomena in different fractional-order non-linear systems.

Analytical research was also important in fractional-order dynamical system because it could present the direct relations between different kinds of solutions with the system parameters. Qi and Xu [26] analyzed the unsteady flow of viscoelastic fluid with the fractional-order derivative Maxwell model. Wahi and Chatterjee [27] studied an oscillator with special fractional-order derivative (p=0.5) and time-delay by averaging method. Chen and Zhu [28,29], Padovan and Sawicki [30], Borowiec et al. [31], and Huang and Jin [32] also investigated different fractional-order systems and presented important results by analytical research. However, these analytical researches were only focused on some special fractional orders, or the fractional-order derivative was simply considered as a special damping force, which may be insufficient in some cases. By the averaging method, Shen et al. studied a linear oscillator [33] and a Duffing oscillator [34] with fractional-order derivative where the fractional order was between 0 and 1, and established the equivalent damping coefficient and the equivalent stiffness coefficient to characterize the effects of the fractional-order derivative on system dynamics.

In this paper, we intend to study the Duffing oscillator with two kinds of fractional-order derivatives, where the fractional-order derivatives are classified based on their range and could cover all the cases. This paper is organized as follow. In Section 2 the primary resonance of the Duffing oscillator with two kinds of fractional-order derivatives is investigated, where two important formulae are presented and the approximately analytical solution is obtained. Additionally, the effects of the parameters in the two fractional-order derivatives on the system damping and stiffness are formulated as the equivalent damping coefficient and the equivalent stiffness coefficient, which is remarkably different from the results in most other existed works. Section 3 presents the steady-state solution, the amplitude–frequency equation, and the stability condition of the steady-state solution. At last, the comparison of the approximately analytical solution with the numerical one is fulfilled in Section 4, and the effects of the parameters in the second kind of fractional-order derivative on the amplitude–frequency equation are also given in this section. Moreover, two special cases, i.e. two appropriate selection methods for the fractional-order coefficient in the second kind of fractional-order derivative are presented, which may be very useful in vibration control engineering.

2. Approximately analytical solution of Duffing oscillator with two kinds of fractional-order derivatives

The considered SDOF Duffing oscillator with two kinds of fractional-order derivatives is shown as

$$m\ddot{x}(t)+kx(t)+c\dot{x}(t)+\alpha_1 x^3(t)+K_1D^{p_1}[x(t)]+K_2D^{p_2}[x(t)]=F\cos(\omega t), \tag{1}$$

where m, k, c, α_1, F, ω are the system mass, linear stiffness coefficient, linear viscous damping coefficient, non-linear stiffness coefficient, excitation amplitude and excitation frequency respectively. In Eq. (1), $K_1D^{p_1}[x(t)]$ is the first kind of fractional-order derivative $x(t)$ to t with the fractional coefficient K_1 ($K_1>0$) and the fractional order p_1, where $2(n-1)\le p_1\le 2n-1$ and n is natural number. $K_2D^{p_2}[x(t)]$ is the second kind of fractional-order derivative of $x(t)$ to t with the fractional coefficient K_2 ($K_2>0$) and the fractional order p_2 ($2n-1\le p_2\le 2n$). These two kinds of fractional-order derivatives could cover all the possibility of the fractional orders, when the fractional orders are limited in real number field. In the next discussion we could find the difference of these two kinds of fractional-order derivatives is remarkable.

There are several definitions for fractional-order derivative, and they are equivalent under some conditions for a wide class of functions. In Caputo's sense, the definition of p order derivative of $x(t)$ to t is

$$D^p[x(t)]=\frac{1}{\Gamma(n-p)}\int_0^t\frac{x^{(n)}(u)}{(t-u)^{p-n+1}}\mathrm{d}u, \tag{2}$$

where $n-1<p<n$, $\Gamma(z)$ is Gamma function satisfying $\Gamma(z+1)=z\Gamma(z)$.

Using the following transformation of coordinates:

$$\omega_0=\sqrt{\frac{k}{m}},\ 2\varepsilon\mu=\frac{c}{m},\ \varepsilon\alpha=\frac{\alpha_1}{m},\ \varepsilon k_1=\frac{K_1}{m},\ \varepsilon k_2=\frac{K_2}{m},\ \varepsilon f=\frac{F}{m}, \tag{3}$$

Eq. (1) becomes

$$\ddot{x}(t)+\omega_0^2x(t)+2\varepsilon\mu\dot{x}(t)+\varepsilon\alpha x^3(t)+\varepsilon k_1D^{p_1}[x(t)]+\varepsilon k_2D^{p_2}[x(t)]$$
$$=\varepsilon f\cos(\omega t), \tag{4}$$

where ω_0 is natural frequency. It should be pointed out that in this transformation, ε, μ, α, k_1, k_2 and f are not dimensionless quantity, and the transformation is only to satisfy the requirement for averaging method formally. The primary resonance means the excitation frequency is close to the natural one, i.e. $\omega\approx\omega_0$. The approximate degree of these two frequencies could be denoted by

$$\omega^2=\omega_0^2+\varepsilon\sigma, \tag{5}$$

where σ is the detuning factor in averaging method [35–37]. Accordingly Eq. (4) could be re-written as

$$\ddot{x}(t)+\omega^2x(t)=\varepsilon\{f\cos(\omega t)+\sigma x(t)-2\mu\dot{x}(t)$$
$$-\alpha x^3(t)-k_1D^{p_1}[x(t)]-k_2D^{p_2}[x(t)]\} \tag{6}$$

Assuming Eq. (6) has the solution as

$$x=a\cos\varphi, \tag{7a}$$

and

$$\dot{x}=-a\omega\sin\varphi, \tag{7b}$$

where the amplitude a and the generalized phase φ ($\varphi=\omega t+\theta$) are slow-varying functions of t. By differentiating Eq. (7a) to t, one could obtain

$$\dot{x}=\dot{a}\cos\varphi-a(\omega+\dot{\theta})\sin\varphi \tag{8}$$