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# Thermodynamic restrictions and wave features of a non-linear Maxwell model

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ABSTRACT

A non-linear rate-type constitutive equation, established by Rajagopal, provides a generalization of the Maxwell fluid. This note embodies such a constitutive equation within the scheme of materials with internal variables thus allowing also for solids with both dissipative and thermoelastic mechanisms. The compatibility with the second law of thermodynamics, expressed by the Clausius–Duhem inequality, is examined and the restrictions on the evolution equations are determined. Next the propagation condition of discontinuity waves is derived, for shock waves and acceleration waves, by regarding the body as a definite conductor. Infinitesimal shock waves and acceleration waves show similar effects. The effective acoustic tensor proves to be the sum of a thermoelastic tensor and a tensor arising from the rate-type equation.



## 1. Introduction

Among the linearized models of viscoelastic materials, those of Boltzmann for solids and Maxwell for fluids are perhaps the simplest, though widely applied, constitutive equations. Both of them can also be viewed as models of materials with fading memory.

In the one-dimensional version, Boltzmann model [1,2] of viscoelastic solid can be expressed by saying that, at any point of the body

$$\sigma(t) = G_0\, \epsilon(t) + \int_{-\infty}^{t} G'(t-\tau)\epsilon(\tau)\, d\tau, \tag{1}$$

where $\sigma$ is the stress and $\epsilon$ is the (infinitesimal) strain. The function

$$G(\xi) = G_0 + \int_0^{\xi} G'(\tau)\, d\tau$$

is called the relaxation function and is assumed to exist on $[0,\infty)$. Moreover, $G_\infty$ denotes the finite positive limit of $G(\xi)$ as $\xi \to \infty$.

The fluid model introduced by Maxwell [3] is given by the equation

$$\sigma + \lambda\dot{\sigma} = \mu\dot{\epsilon}, \quad \lambda,\mu > 0,$$

a superposed dot denoting the time derivative. Integration gives

$$\sigma(t) = \frac{\mu}{\lambda}\int_{-\infty}^{t} \exp(-(t-\tau)/\lambda)\dot{\epsilon}(\tau)\, d\tau, \tag{2}$$

thus showing that the material constant $\lambda$ can be viewed as a relaxation time. The constant $\mu$ has the units of viscosity and is referred to as the viscosity of the fluid. The fluid character of the Maxwell model is shown by the vanishing of $\sigma$ generated by a constant strain $\epsilon$. Also, an integration by parts of (2) gives formally (1) with

$$G_0 = \frac{\mu}{\lambda}, \quad G'(\xi) = \frac{\mu}{\lambda^2}\exp(-\xi/\lambda).$$

In such a case, though,

$$G_\infty = \frac{\mu}{\lambda} + \frac{\mu}{\lambda}\int_0^{\infty} \exp(-\xi/\lambda)\, d\xi = 0.$$

Concerning fluids, generalizations of the Maxwell model were developed by Rajagopal and Srinivasa [4] and Rao and Rajagopal [5] essentially by letting the viscosity $\mu$ and the relaxation time $\lambda$ depend on the invariants of the Cauchy–Green stretch tensor.

In a recent paper [6], Rajagopal generalizes the Maxwell model by viewing the body as being made up of two components. One component behaves elastically like a generalized spring in that it shows a non-linear relationship between stress and strain. The other component behaves like a viscous dashpot and the relation between stress and strain rate is non-linear. The approach is uncommon in that it involves non-linear implicit relationships between kinematical quantities and the stress instead of the stress as a (non-linear) function of the kinematical quantities.

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Motivated by the interest in the non-linear constitutive equation so established, the purpose of this paper is twofold. First, to investigate the compatibility with the second law of thermodynamics of non-linear differential (or rate-type) constitutive equations. In so doing we look for a further generalization so that both solids and fluids are modeled by constitutive equations of the type established in [6]. It is worth pointing out that a thermodynamic framework for rate-type constitutive equations is developed in [4], the thermodynamic requirement being an appropriate criterion of maximum rate of dissipation. In this paper, instead, the thermodynamic analysis is based on the Clausius–Duhem inequality as the requirement of the second law and the rate-type equation is framed within the scheme of materials with internal variables [7–9]. The second fold is to examine discontinuity waves and to determine the effects of constitutive functions on propagation properties. The interest of this analysis is given by the non-linearity of the model and the rate-type constitutive equation.

Though some comments, given in the next section, may suggest the way toward more general models, it is of interest to see how the non-linear constitutive equation given in [6] may be framed within the Eulerian description with the second law expressed by the Clausius–Duhem inequality.

## 2. Balance equations and the non-linear model

Let $\Omega$ be the time-dependent region occupied by the body and denote by $\mathbf{x}$ the position vector relative to a fixed origin. The symbol $\nabla$ denotes the gradient operator, with respect to $\mathbf{x}$. Let $\mathbf{u}(\mathbf{x},t)$ be the displacement function, on $\Omega\times\mathbb{R}$, and $\boldsymbol{\epsilon}=\frac{1}{2}[\nabla\mathbf{u}+(\nabla\mathbf{u})^T]$ the strain, the superscript $T$ denoting transpose. The stress tensor is denoted by $\mathbf{T}$ and the velocity by $\mathbf{v}$. A superposed dot denotes the material time derivative, $\dot{}=\partial_t+\mathbf{v}\cdot\nabla$ and Sym is the set of symmetric (second-order) tensors.

Consistent with [6], we have in mind a scheme which is a linearized approximation with respect to $\boldsymbol{\epsilon}$. Hence, as with the linearized theory of viscoelasticity, we regard the mass density $\rho$ as a function of $\mathbf{x}\in\Omega$ and disregard the time dependence. The continuity equation is then omitted. Also we let $\mathbf{T}\cdot\dot{\boldsymbol{\epsilon}}$ be (the approximation of) the Cauchy stress power. The balance of linear momentum and energy are written in the form

$$\rho\dot{\mathbf{v}}=\nabla\cdot\mathbf{T}+\rho\mathbf{b}, \tag{3}$$

$$\rho\dot{e}=\mathbf{T}\cdot\dot{\boldsymbol{\epsilon}}+\rho r-\nabla\cdot\mathbf{q}, \tag{4}$$

where $\mathbf{b}$ is the body force (per unit mass), $e$ is the internal energy density, $r$ is the heat supply, $\mathbf{q}$ is the heat flux vector.

Let $\eta$ be the entropy density and $\theta$ the absolute temperature. The constitutive equations are said to be physically admissible if they comply with the entropy inequality

$$\rho\dot{\eta}\geq\frac{\rho r}{\theta}-\nabla\cdot(\mathbf{q}/\theta) \tag{5}$$

for any process on $\Omega\times\mathbb{R}$ satisfying the balance equations. In terms of the free energy

$$\psi=e-\theta\eta,$$

upon substitution of $\rho r-\nabla\cdot\mathbf{q}$ from the energy equation, inequality (5) becomes the Clausius–Duhem inequality

$$-\rho(\dot{\psi}+\eta\dot{\theta})+\mathbf{T}\cdot\dot{\boldsymbol{\epsilon}}-\frac{1}{\theta}\mathbf{q}\cdot\nabla\theta\geq 0. \tag{6}$$

To establish the appropriate class of constitutive models we need a variable $\boldsymbol{\sigma}$, eventually related to $\mathbf{T}$, characterized by an evolution equation governed by $\boldsymbol{\epsilon}$. Let

$$\mathbf{f},\mathbf{g}:\mathrm{Sym}\to\mathrm{Sym}.$$

The generalized model of [6] is characterized by the constitutive equation

$$\dot{\mathbf{f}}(\boldsymbol{\sigma})+\mathbf{g}(\boldsymbol{\sigma})=\dot{\boldsymbol{\epsilon}} \tag{7}$$

The function $\mathbf{f}$ is assumed to be twice continuously differentiable whereas $\mathbf{g}$ is only required to be continuous.

By (7) we have

$$\mathbf{f}_{\boldsymbol{\sigma}}\dot{\boldsymbol{\sigma}}+\mathbf{g}=\dot{\boldsymbol{\epsilon}},$$

where $\mathbf{f}_{\boldsymbol{\sigma}}$ is the derivative of $\mathbf{f}$, with respect to $\boldsymbol{\sigma}$, and

$$\mathbf{f}_{\boldsymbol{\sigma}}:\mathrm{Sym}\to\mathrm{Sym}.$$

We assume that $\mathbf{f}_{\boldsymbol{\sigma}}$ is invertible and let $\mathbf{A}=(\mathbf{f}_{\boldsymbol{\sigma}})^{-1}$. Hence we have

$$\dot{\boldsymbol{\sigma}}=\mathbf{A}(\boldsymbol{\sigma})[\dot{\boldsymbol{\epsilon}}-\mathbf{g}(\boldsymbol{\sigma})]. \tag{8}$$

By (8) we can say that $\dot{\boldsymbol{\sigma}}$ is a non-linear function of $\boldsymbol{\sigma}$ and linear in $\dot{\boldsymbol{\epsilon}}$.

In [6], $\boldsymbol{\sigma}$ is identified with the Cauchy stress. For a greater generality, here $\boldsymbol{\sigma}$ need not be equal to $\mathbf{T}$. As a consequence, we will find a class of constitutive equations that characterize Maxwell-type solids while $\mathbf{T}=\boldsymbol{\sigma}$ applies to fluids.

### 2.1. Comments on the mathematical model

Eq. (7) involves the material time derivative, of $\boldsymbol{\epsilon}$ and $\mathbf{f}$, which is not frame indifferent. As in [6], and consistent with the Eulerian description, we regard (7) as a linearized approximation and that is why we are content with the material time derivative.

The same view about the linearized approximation motivates the power of the stress as $\mathbf{T}\cdot\dot{\boldsymbol{\epsilon}}$ instead of the exact term $\mathbf{T}\cdot\mathbf{D}$, $\mathbf{D}$ being the stretching tensor.

Non-linear models free from the restriction of linearized approximation may be addressed in various ways. A line of approach is based on non-linear terms in the Lagrangian description, see e.g. [4,5,10]. Other fluid models involve rate-type constitutive equations such as Oldroyd-B fluid, Rivlin–Ericksen fluid and Burgers fluid possibly with non-linear terms [11–13]. The investigation of such models, within a thermodynamic setting based on the Clausius–Duhem inequality, requires a fully non-linear scheme and a Lagrangian description as is done in [9].

## 3. Thermodynamic restrictions

The tensor function $\boldsymbol{\sigma}$, on $\Omega\times\mathbb{R}$, is regarded as an internal variable whose evolution is governed by (8), for any point in $\Omega$. Indeed, for any values of $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}$, Eq. (8) provides $\dot{\boldsymbol{\sigma}}$. Also, we may regard $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}$ as independent values whereas $\dot{\boldsymbol{\sigma}}$ is determined by (8).

We let

$$\Gamma=(\theta,\boldsymbol{\epsilon},\boldsymbol{\sigma},\dot{\boldsymbol{\epsilon}},\nabla\theta)$$

be the set of (independent) variables. However, $\dot{\boldsymbol{\sigma}}$ depends in fact only on $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\epsilon}}$ through (8) whereas $\psi,\eta,\mathbf{T},\mathbf{q}$ are assumed to be independent of $\dot{\boldsymbol{\epsilon}}$. If, instead, we start from the assumption that $\psi,\eta,\mathbf{T},\mathbf{q}$ depend on $\Gamma$ as a whole then we see at once that the Clausius–Duhem inequality rules out the dependence of $\psi$ and $\eta$ on $\dot{\boldsymbol{\epsilon}}$ but $\mathbf{T}$ and $\mathbf{q}$ may depend on $\dot{\boldsymbol{\epsilon}}$, which does not allow the following detailed conclusions.

Time differentiation of $\psi$ and substitution of (8) in the Clausius–Duhem inequality (6) provide

$$-\rho(\psi_\theta+\eta)\dot{\theta}+(-\rho\psi_{\boldsymbol{\epsilon}}+\mathbf{T})\cdot\dot{\boldsymbol{\epsilon}}-\rho\psi_{\boldsymbol{\sigma}}\cdot\mathbf{A}(\dot{\boldsymbol{\epsilon}}-\mathbf{g})-\rho\psi_{\nabla\theta}\cdot\overline{\nabla\theta}^{\,\cdot}$$
$$-\frac{1}{\theta}\mathbf{q}\cdot\nabla\theta\geq 0, \tag{9}$$