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A theoretical analysis of unsteady incompressible flows for time-dependent constitutive equations based on domain transformations

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ABSTRACT

In this paper, we propose an analysis that allows calculation of kinematic histories in unsteady problems of continuum mechanics, in relation to the use of memory-integral constitutive equations. Such cases particularly concern flow conditions of processing rheology, requiring evaluation of strain or deformation rate tensors, for viscoelastic incompressible fluids as polymers. In two- and three-dimensional cases, we apply concepts of the stream-tube method (STM) initially given for stationary conditions, where unknown local or global mapping functions are defined instead of classic velocity–pressure formulations, leading to consider the flow parameters in domains where the streamlines and trajectories are parallel straight lines. The approach enables us to provide accurate formulae for evaluating the kinematics histories that can be used later for computing the stresses for a given memory-integral model.

1. Introduction

Time-dependent constitutive equations are generally expressed under the general tensorial form

$$\underline{\underline{\tau}}(t) = \underline{\underline{H}}^{t}_{-\infty}[\underline{\underline{K}}_i(t,t')] \tag{1}$$

where $\underline{\underline{\tau}}(t)$ denotes the extra-stress tensor at time t, $\underline{\underline{H}}^{t}_{-\infty}(t)$ is a functional tensor of kinematic tensors $\underline{\underline{K}}_i(t,t')$ $(i = 1,\ldots,I_0)$ for times $t' \leqslant t$. Expressing such quantities requires evaluation of time evolution of particles on their pathlines. Eq. (1) is generally used on the form of memory-integral expressions [1–8]. Representations of the stress tensor are given with rate constitutive equations as corotational laws [3,5] expressed with the rate-of-deformation tensor $\underline{\underline{D}}(t) = (1/2)(\underline{\underline{\nabla V}} + \underline{\underline{\nabla V}}^T)$ in corotational reference frames and, more widely, with codeformational models [1,2,4–8] involving deformation tensors as the respective Cauchy and Finger tensors $\underline{\underline{C}}_t(t')$ and $\underline{\underline{B}}(t')$ derived from the deformation tensor $\underline{\underline{F}}_t(t')(t' \leqslant t)$. An example of this type is the popular K–BKZ model (e.g. [5]) expressed by the equation

$$\underline{\underline{\tau}}(t) = \int_{-\infty}^{t} \left[\frac{\partial U}{\partial I_1} \underline{\underline{B}}_t(t') - \frac{\partial U}{\partial I_2} \underline{\underline{C}}_t(t') \right] dt'' \tag{2}$$

where U denotes a potential of the invariants $I_1 = \mathrm{tr}(\underline{\underline{B}})$ and $I_2 = \mathrm{tr}(\underline{\underline{C}})$.

Particularly, few numerical papers have concerned time-dependent constitutive equations in non-stationary flow cases [9]. Since the pathlines do not pass through the mesh points of grids defined in numerical applications, significant accuracy problems arise in such cases. Together with fundamental interests, the study of unsteady flows of materials obeying memory-integral viscoelastic constitutive equations is considered in relation to numerous industrial processes of polymers, food fluids,... as extrusion and injection molding and general two- and three-dimensional cases. At the moment, the numerical simulations involve coupling of sophisticated rheological models and conservation laws in order to determine flow characteristics in geometries that are also complex. However, difficulties arise owing to the convective character of the viscoelastic constitutive equations, since the stress state in a fluid element depends on its strain history. This implies the use of upwind discretising schemes that require robust and efficient stability techniques ensuring accuracy and convergence of the solving procedures.

Focusing on incompressible materials, the purpose of the present paper is to extend possibilities of the stream-tube method (STM) to evaluation of kinematic quantities in unsteady flows, where streamlines and pathlines are not identical, in relation to time-dependent constitutive equations. The approach is considered through streamline and pathline determination at every time t, starting from the rest state or from stationary conditions for fluid particles of a domain Ω of boundary Γ given an initial time t_0.

In STM analysis of stationary cases [10,11] where the concepts of streamlines are used, the fluid dynamics problem is reformulated by using global or local transformations of the physical domain, that must be considered in the governing equations instead of

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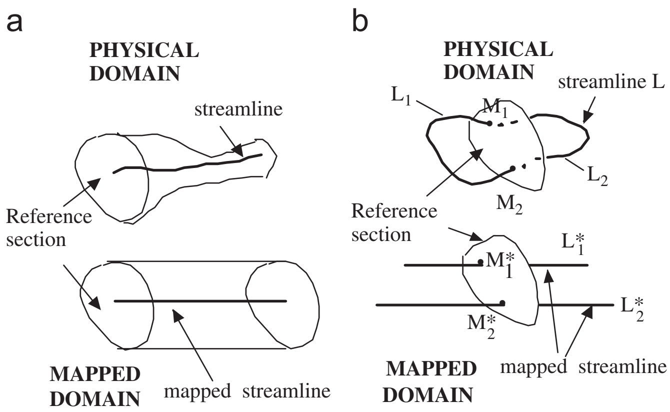

Fig. 1. Transformation of streamlines in stream-tube method: (a) global transformation for an open streamline and (b) local transformation for a closed streamline divided into two branches.

classical primary unknowns as the velocity. Together with the pressure, the local or global unknown transformation functions of the physical domain, ensuring that the mapped streamlines are parallel and straight, must be determined from the governing equations, written with variables of the mapped domains where the computations are performed. It should be pointed out that, in STM, flow with open or closed streamlines can be considered, according to transformations into rectilinear lines summarized in Fig. 1a and b [11,12]. From a theoretical viewpoint, STM has allowed to propose simple and precise formulae to evaluate kinematic and strain histories as already provided in previous papers [14,15] under stationary conditions, for fluids obeying time-dependent constitutive equations. In practice, when performing STM simulations related to steady configurations, the nodes are naturally set on the rectilinear streamlines allowing an optimal accuracy. The calculation of stresses along the mapped closed streamlines avoids problems of numerical diffusion that can occur, in classical methods, from diffusion related to the singularity from the center of rotation of the vortex zone that cannot be improved by convective stabilization techniques. In unsteady cases, such accuracy problems are increased.

Non-stationary flows have been already considered using STM analysis in order to compute start-up and pulsating flows between concentric and eccentric cylinders with purely viscous non-Newtonian constitutive equations [13]. In these cases, streamlines have been determined versus time without referring to the strain history. Concerning approaches related to calculation of strain histories in steady flows, an analysis has been proposed by Adachi [16,17], from concepts referred to Protean coordinates [18], where one coordinate is a streamline. Starting from this study, Clermont [16] applied the theoretical elements of the STM to evaluate strain rates and strain histories in two- and three-dimensional steady situations, for flows involving only open streamlines. In STM, this latter case corresponds to the use of one or two global transformation functions, in two and three dimensions, respectively.

We wish to develop theoretical tools of the STM for determining trajectories and histories for moving particles, according to open and closed streamlines of two- and thee-dimensional non-stationary flows. In this paper, we first recall briefly the background on global and local transformations related to a bounded flow domain Ω, in steady conditions. Then, we go further by proposing an approach towards evaluation of the kinematics history for non-stationary flows, based on elements previously provided for streamlines, in steady conditions, that allows computation of pressure and mapping functions for streamlines and trajectories the transformed of which are parallel straight lines, in rectangles or parallelepipeds.

Global one-to-one transformations $T_{(t)}$ of a flow domain Ω with open streamlines are related to the use of one or two mapping functions, in two- and three-dimensional cases, respectively [9]. At time t, the corresponding equations, expressed in Cartesian coordinates X^i ($X^1 = x$, $X^2 = y$, $X^3 = z$) for the physical domain Ω and q^i ($q^1 = X$, $q^2 = Y$, $q^3 = Z$) for its transformed Ω^* are given by

$$T_{(t)} : \Omega^* \to \Omega : (X, Z) \to (x, z) \tag{3}$$

such that

$$x = f(X, Y); \quad z = Z \tag{4}$$

in the planar case and, in the three-dimensional case

$$T_{(t)} : \Omega^* \to \Omega : (X, Y, Z) \to (x, y, z) \tag{5}$$

with

$$x = f(X, Y, Z); \quad y = g(X, Y, Z); \quad z = Z. \tag{6}$$

For the above transformations, we define an upstream reference section S_{ref} at $z = z_{ref}$ where the kinematics are known, mapped into a section S^*_{ref} of Ω^*, identical in shape to, S_{ref} such that

$$x = f(X, Y, Z_{ref}); \quad y = g(X, Y, Z_{ref}); \quad z_{ref} = X_{ref}. \tag{7}$$

The mapped domain Ω^* is a straight cylinder of basis S^*_{ref} , of transformed streamlines parallel to the mean flow direction.

Similar relations can be obtained with cylindrical coordinates (r, θ, Z) as

$$r = f(R, \theta, Z); \quad \theta = g(R, \Theta, Z); \quad z = Z. \tag{8}$$

In the cases investigated, the existence of the reference section S_{ref} at $z = z_{ref}$, where the kinematics are known, is assumed [9] to compute the velocity field in the flow domain. The incompressibility condition is automatically verified by the formulation, using expressions involving stream functions.

When flow vortices are expected in domain Ω, local mappings $T_{m_{(t)}} : \Omega^*_m \to \Omega_m$ are adopted, using domain decomposition such that $\Omega = \cup_{m=1}^{m=M} \Omega_m$. When using Cartesian coordinates X^i ($X^1 = x$, $X^2 = y$, $X^3 = z$) in domain Ω , we define a coordinate system ξ^j($\xi^1 = X$, $\xi^2 = Y$, $\xi^3 = s$) in a sub-domain Ω^*_m, where the streamlines are parallel segments. The variable s is defined as the length of a segment of streamline in Ω^*_m, being zero at a reference section S_m, with two branches of the streamline (Fig. 1b). Assuming, in domain Ω^*_m, the existence of a reference section S^*_m identical in shape to the reference section S_m of the sub-domain Ω_m of Ω, we can write the correspondence between sub-domain Ω_m and a local domain Ω^*_m according to the following relations [11,12].

$$x = \alpha_m(X, Y, s); \quad y = \beta_m(X, Y, s); \quad z = \gamma_m(X, Y, s). \tag{9}$$

For each domain Ω^*_m we consider a kinematic function ϕ^*_m related to the reference section S^*_m, leading to write the velocity components in terms of ϕ^*_m and variables ξ^j [12]. The Jacobian $\Delta = |\partial(X^i)/\partial(\xi^j)|$, assumed to be non-zero, is given by the equation

$$\Delta = \alpha'_X(\beta'_Y\gamma'_S - \beta'_S\gamma'_Y) - \beta'_X(\alpha'_Y\gamma'_S - \gamma'_Y\alpha'_S) + \gamma'_X(\alpha'_Y\beta'_S - \beta'_Y\alpha'_S). \tag{10}$$

To compute the flow field, the local mapping approach requires to write compatibility equations at the common boundaries of the sub-domains that define the total domain Ω together with the classic conservation equations.