International Journal of Non-Linear Mechanics 46 (2011) 958–970

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Remarks on the solution of extended Stokes' problems

Giorgio Riccardi *

Department of Aerospace and Mechanical Engineering, Second University of Naples, via Roma, 29-81031 Aversa (CE), Italy

ARTICLE INFO

Article history:
Received 10 August 2010
Received in revised form
19 March 2011
Accepted 5 April 2011
Available online 1 May 2011

Keywords:
Stokes' flow
First and second Stokes' problems
Wall stress
Newtonian fluid
Analytical solution

ABSTRACT

The analytical solutions of first and second Stokes' problems are discussed, for infinite and finite-depth flows of a Newtonian fluid in planar geometries. Problems arising from the motion of the wall as a whole (one-dimensional flows) as well as of only one half of the wall (two-dimensional) are solved and the wall stresses are evaluated.

The solutions are written in real form. In many cases, they improve the ones in literature, leading to simpler mathematical forms of velocities and stresses. The numerical computation of the solutions is performed by using recurrence relations and elementary integrals, in order to avoid the evaluation of integrals of rapidly oscillating functions.

The main physical features of the solutions are also discussed. In particular, the steady-state solutions of the second Stokes' problems are analyzed by separating their "*in phase*" and "*in quadrature*" components, with respect to the wall motion. By using this approach, stagnation points have been found in infinite-depth flows.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

The analytical solution of Stokes problems for a Newtonian fluid in a planar geometry is here revised, by following the seminal paper of Liu [1]. A fluid region is bounded by a rigid wall, which moves with a prescribed velocity having fixed direction, parallel to the wall. The fluid and the wall are at rest at the initial time. By following the literature, wall velocities constant (first Strokes' problem) or periodical (second) in time will be assumed. Moreover, flows in which the wall moves as a whole (one-dimensional) and half wall moves, while the other one is kept fixed, (two-dimensional) will also be investigated. Finally, the depth of the fluid region will be assumed infinite or finite. In these latter kinds of flow, a free surface is assumed to bound the fluid region.

The solution of the first Stokes' problem in an infinite-depth flow has a well known analytical structure, related to the complementary (real) error function. Solutions of the second problem in an infinite-depth flow have been discussed in [2–4] and in many other papers. They are usually written in terms of error functions of complex arguments, because in the corresponding real forms integrands containing oscillatory functions appear, the numerical integration of which can lead to severe errors [5].

Recently, these results have been reconsidered in the framework of two-dimensional flows. In the paper [6] the steady states have been found, while Liu [1] generalizes these solutions, by giving also the transient contributions. The effects of side walls on the Stokes flow on a planar wall have been recently investigated in [7]. Besides the first and second Stokes problems, the flows induced by a constant accelerating plate and by a plate that applies a constant stress are also investigated. This important paper opens the way to the comparison with experiments, where effects of side walls are rarely negligible.

Despite the subject is a quite old one [8], many issues about analytical solutions and their numerical computation appear to be improved, in particular for two-dimensional flows. The present paper is an attempt to fill some of these lacks. It is organized as follows. In Section 2, the solutions of one-dimensional first and second problems are briefly discussed, then they are extended to the finite-depth case in Section 3. The solution of two-dimensional problems is then faced, for infinite (Section 4) and finite-depth (Section 5) flows. Finally, conclusions are offered in Section 6.

2. One dimensional infinite-depth flows

A Newtonian fluid having kinematical viscosity ν fills the half space $y>0$, bounded by a solid wall at $y=0$. Initially ($t \le 0$), fluid and wall are at rest. The wall starts to move at time $t=0^+$ with a given velocity (say q), directed along the axis x. The resulting fluid velocity (u) is assumed to be directed along x and to depend on y and t, only. As well known, this flow is described by Stokes' problem:

$$\begin{cases} \partial_t u = \nu \partial^2_{yy} u, \\ u(0,t) = q(t), \quad u(+\infty,t) \equiv 0, \\ u(y,0) \equiv 0, \end{cases}$$

the solution of which is easily found in terms of Laplace transform in time ($q^{(L)}$ and $u^{(L)}$ are the transformed functions of q and u,

* Tel.: +39 081 5010283; fax: +39 081 5010264.
E-mail address: giorgio.riccardi@unina2.it

0020-7462/$ - see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijnonlinmec.2011.04.010

respectively):

$$u^{(L)}(y,s) = \exp(-\beta y)q^{(L)}(s). \tag{1}$$

Here, the complex variable s has a positive real part and $\beta = \sqrt{s/\nu}$ (the principal branch of the root is used). In the following, two different wall velocities will be considered: constant, i.e. $q(t) \equiv 0$ as $t < 0$ and $q(t) \equiv u_0$ as $t > 0$, which leads to the *first* Stokes' problem and periodical, i.e. $q(t) = u_0 \cos(\omega t + \theta)$ as $t > 0$, corresponding to the *second* Stokes' problem.

2.1. First Stokes' problem

The solution of this classical problem is here summarized, for later convenience. The Laplace transform of the wall velocity is

$$q^{(L)}(s) = \frac{u_0}{s}, \tag{2}$$

so that the time derivative of the non-dimensional velocity $U_1 = u_1/u_0$ (non-dimensional quantities will be indicated by capital symbols, while the subscript 1 refers to the first solution of the present paper) is obtained through a Laplace antitransform of the general solution (1):

$$\partial_t U_1 = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} ds \exp(ts - y\beta) =: F_1, \tag{3}$$

μ being a suitable positive real number. The function F_1 is calculated by applying Cauchy's theorem to the integral of $\exp(ts - y\beta)/(2\pi i)$ on the path of Fig. 1a and then by performing the limit as $M \to +\infty$. The two resulting integrals are evaluated along the lower and upper paths of Fig. 1b: it is found that their sum gives $\sqrt{\pi}$. As a consequence, F_1 assumes the following form:

$$F_1(y,t) = \frac{1}{2\sqrt{\pi\nu}} y t^{-3/2} \exp\left(-\frac{y^2}{4\nu t}\right). \tag{4}$$

Once it is inserted in Eq. (3), an integration in time leads to the classical solution:

$$U_1(Y,T) = \frac{2}{\sqrt{\pi}} \int_{Y/(2\sqrt{T})}^{+\infty} d\eta\, e^{-\eta^2} = \operatorname{erfc}\left(\frac{Y}{2\sqrt{T}}\right), \tag{5}$$

in which lengths and times are non-dimensionalized with ν/u_0 and ν/u_0^2, respectively. It can be observed that the velocity (5) depends on Y and T through the time-rescaled variable $Y' = Y/(2\sqrt{T})$: written in terms of a function of Y', the above velocity will be indicated hereafter by $U_1'(Y')$. The wall stress w_1 follows in non-dimensional form as $W_1 = w_1/(\rho u_0^2)$, ρ being the fluid density. By using the solution (5), one obtains:

$$W_1(T) = -1/\sqrt{\pi T}. \tag{6}$$

2.2. Second Stokes problem

In the second Stokes' problem, the Laplace transform of the wall velocity is

$$q^{(L)}(s) = \frac{u_0}{2}\left(\frac{e^{-i\theta}}{s + i\omega} + \frac{e^{+i\theta}}{s - i\omega}\right), \tag{7}$$

so that the general solution (1) is specified in the following one:

$$U_2 = \frac{1}{2}\left\{ e^{-i(\omega t + \theta)} \underbrace{\frac{1}{2\pi i}\int_{\mu - i\infty}^{\mu + i\infty} ds \frac{\exp[t(s + i\omega) - y\beta]}{s + i\omega}}_{H_2^+} + e^{+i(\omega t + \theta)} \underbrace{\frac{1}{2\pi i}\int_{\mu - i\infty}^{\mu + i\infty} ds \frac{\exp[t(s - i\omega) - y\beta]}{s - i\omega}}_{H_2^-} \right\}. \tag{8}$$

The time derivatives of the functions $H_2^\pm$ are easily evaluated in terms of F_1, indeed: $\partial_t H_2^\pm = \exp(\pm i\omega t)F_1$. Once the proper form of the function F_1 (4) is inserted into the above relations and they are integrated in time, one obtains:

$$H_2^\pm(y,t) = H_2^\pm(y,0) + \frac{y}{2\sqrt{\pi\nu}} \int_0^t d\tau\, \tau^{-3/2} \exp\left(\pm i\omega\tau - \frac{y^2}{4\nu\tau}\right). \tag{9}$$

Notice that, in order to have $u_2(y,0) \equiv 0$ for any initial phase θ, $H_2^\pm(y,0)$ must vanish, as it can be also proved by integrating along the path of Fig. 1a their definitions (8) evaluated in $t=0$. The functions $H_2^\pm$ (9) with $H_2^\pm(y,0) \equiv 0$ are then inserted into the formula (8) and the non-dimensional quantities $T = \omega t$ and $Y = y(\omega/\nu)^{1/2}$ are used, according to [4]. In this way, the solution:

$$U_2(Y,T) = \frac{2}{\sqrt{\pi}} \int_{Y/(2\sqrt{T})}^{+\infty} d\eta\, e^{-\eta^2} \cos\left(T + \theta - \frac{Y^2}{4\eta^2}\right) \tag{10}$$

follows. This solution is the real form of the one in [3] for $\theta = 0$ and $\pi/2$ and of the solution in [4].

The numerical evaluation of the solution (10) is not a trivial task, due to the presence of $1/\eta^2$ in the argument of the trigonometric function. Numerical integration schemes lead to

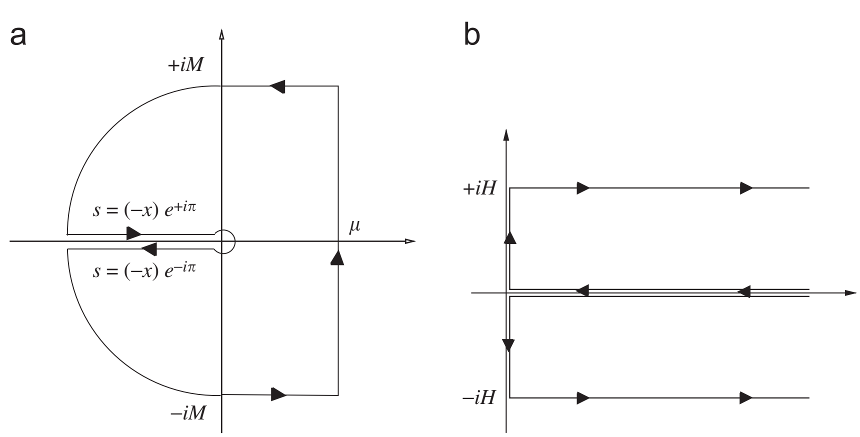

Fig. 1. Integration paths in the plane of s: for the evaluation of the function F (a) and of the integrals $I_{1,2}$.

Download English Version:

https://daneshyari.com/en/article/785176

Download Persian Version:

https://daneshyari.com/article/785176

Daneshyari.com