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Synge's concept of stability applied to non-linear normal modes

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Abstract

Synge's concept [J.L. Synge, On the geometry of dynamics, Philos. Trans. R. Soc. London, Ser. A 226 (1926) 33–106] of stability is introduced and shown to be equivalent to the orbital stability in holonomic conservative systems of two-degrees-of-freedom. This furnishes an analytical tool to study the orbital stability in strongly non-linear systems. This concept is shown to be applicable to the stability analysis of non-linear normal modes, for which Liapunov's first method generally fails. Integrally related numbers are found such that, if the ratio of linear natural frequencies is close to one of the numbers, then a normal mode may lose stability at a small amplitude. These numbers depend on the symmetry or asymmetry of system with respect to the origin of the configuration space. Some examples are given to demonstrate the stability analysis of the normal modes and to verify the integrally related numbers.

Keywords: Synge's concept of stability; Nonlinear normal mode; Orbital stability

1. Introduction

The stability of periodic motions is of interest in holonomic, conservative non-linear systems of two-degrees-of-freedom. The response in free vibration as well as forced response is greatly influenced by the stability or instability of periodic motions present in the system. Unstable motions are sometimes ignored with the view that the response is physically not realized. However, two important phenomena are observed by the loss of stability [1–4]; that is, (i) a stable coupled mode (or bifurcating mode) is formed, giving rise to a stable coupled-mode response under a single mode excitation, and (ii) at large forcing amplitudes an orbit close to the unstable mode is attracted to a strange attractor, leading to chaotic responses.

It is well-known that every motion of a simple pendulum is unstable in the sense of Liapunov because the period varies with amplitude. Similarly, a periodic motion in a multi-degree-of-freedom system is generally unstable in the sense of Liapunov. For this reason, orbital stability was proposed by Poincaré. The orbital stability involves the Poincaré map. Since the motion remains in a three-dimensional energy manifold, the local surface of section is two dimensional. The Poincaré map is usually obtained by numerical computations.

Synge [5] studied dynamics in the view of Riemannian geometry and proposed several concepts of stability. One is closely related with orbital stability. His stability equation is expressed as a single second-order differential equation with periodic coefficients. This leads to a two-dimensional phase space, reminiscent of the two-dimensional Poincaré section. It will be shown that his concept is equivalent to orbital stability. Therefore, his concept furnishes an analytical tool to study the orbital stability in strongly non-linear systems.

Synge's concept is applied to the stability analysis of non-linear normal modes. To this end, the solution of the normal modes is required beforehand. The normal modes are defined on the basis of Liapunov's existence theorem [6–9]; i.e., two normal modes exist in the neighborhood of a stable equilibrium configuration. A normal mode is written in Fourier series, and may be approximated by the first or a few harmonic terms in each coordinate. By the method of harmonic balance, the solution may be obtained. By substituting this solution into Synge's stability equation, the stability chart can be constructed to determine the stability of normal modes.

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The conditions are investigated for the stability loss of normal modes at small amplitude. It is shown that if the ratio of linear natural frequencies is not integrally related, then every normal mode is stable at small amplitude and may lose stability as the amplitude increases. These integrally related numbers are derived for the system having or not having symmetry with respect to the origin of the configuration space.

When Liapunov's first method is applied to the stability analysis of normal modes, the system of variational equations is generally expressed by two coupled second-order differential equations with periodic coefficients. It is practically impossible to construct the stability chart.

2. Synge's concept of stability

Consider a holonomic conservative system in which the kinetic energy T and potential energy V are written as

$$T=\frac{1}{2}\sum_{ij=1}^{2} m_{ij}(\mathbf{x})\dot{x}_i\dot{x}_j=\frac{1}{2}\langle \mathbf{M}(\mathbf{x})\dot{\mathbf{x}},\dot{\mathbf{x}}\rangle,$$
$$V=V(\mathbf{x}),\quad K+V=h, \tag{2.1}$$

where $\boldsymbol{x}=(x_1,x_2)$ and $\dot{\boldsymbol{x}}=(\dot{x}_1,\dot{x}_2)$ are the generalized coordinates and velocities, respectively, $\boldsymbol{M}(\boldsymbol{x})=(m_{ij}(\boldsymbol{x}))$ is the inertia matrix, h the total energy, and the symbol $\langle,\rangle$ denotes the inner product of two vectors, as will be used throughout. For given h, the motion remains in the region given by

$$\Gamma(h)=\left\{\mathbf{x}\in R^2\,|\,h-V(\mathbf{x})\geqslant 0\right\}. \tag{2.2}$$

Assume that $\Gamma(h)$ is bounded. The interior is denoted by $\operatorname{int}\Gamma(h)$, and the boundary by $\partial\Gamma(h)$.

Synge [5] studied dynamics in the view of Riemannian geometry for which the line element $\mathrm{d}s$ is defined by

$$\mathrm{d}s^2=\langle \mathbf{M}\,\mathrm{d}\mathbf{x},\mathrm{d}\mathbf{x}\rangle \tag{2.3}$$

and the angle φ between two vectors, $\boldsymbol{a}=(a_1,a_2)$ and $\boldsymbol{b}=(b_1,b_2)$, is given by

$$\cos\varphi=\frac{\langle \mathbf{Ma},\mathbf{b}\rangle}{\sqrt{\langle \mathbf{Ma},\mathbf{a}\rangle}\sqrt{\langle \mathbf{Mb},\mathbf{b}\rangle}}. \tag{2.4}$$

He defined the disturbance $\boldsymbol{\beta}$ between configurations of the periodic trajectory C and a perturbed trajectory C^* by the condition that $\boldsymbol{\beta}$ is orthogonal to C, as shown in Fig. 1(a). By use of tensor analysis he derived the equation governing the magnitude β, for fixed h,

$$\ddot{\beta}+Q(t)\beta=0,$$
$$Q(t)=Kv^2+3\kappa^2v^2+\sum V_{ij}n_in_j, \tag{2.5}$$

where v is the velocity of C, κ the curvature of C, $V_{ij}=\partial^2V/\partial x_i\partial x_j$, $\boldsymbol{n}=(n_1,n_2)$ the unit normal to C, and K Gaussian curvature written as

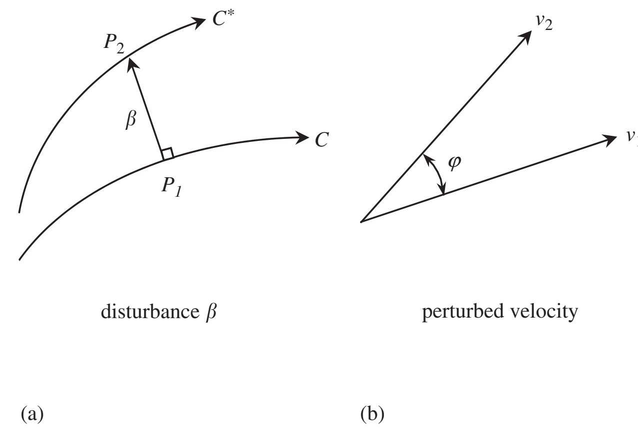

Fig. 1. Definition of disturbance. (a) Disturbance β, (b) perturbed velocity.

$$K=\frac{1}{2\sqrt{m}}\left\{\frac{\partial}{\partial x_1}\left(\frac{m_{12}}{m_{11}\sqrt{m}}\frac{\partial m_{11}}{\partial x_2}-\frac{1}{\sqrt{m}}\frac{\partial m_{22}}{\partial x_1}\right)\right.$$
$$\left.-\frac{\partial}{\partial x_2}\left(\frac{2}{\sqrt{m}}\frac{\partial m_{12}}{\partial x_1}-\frac{m_{12}}{m_{11}\sqrt{m}}\frac{\partial m_{11}}{\partial x_1}-\frac{1}{\sqrt{m}}\frac{\partial m_{11}}{\partial x_2}\right)\right\}, \tag{2.6}$$

where $m=m_{11}m_{22}-m_{12}^2$. Then C is said to be stable in the kinematico-statical sense (or simply S-stable) if the value of β, for every solution of Eq. (2.5), is permanently small.

By use of the definition of $\boldsymbol{\beta}$, the time derivative $\dot{\beta}$ may be geometrically derived. Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ be the velocities of C and C^* at P_1 and P_2, respectively, as shown in Fig. 1(b), in which the line P_1P_2 is orthogonal to C. Then it is clear that

$$\dot{\beta}=v_2\sin\varphi, \tag{2.7}$$

where φ is the angle between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Remark 1. The Riemannian metric given by Eq. (2.3) is equivalent to the Euclidean norm; i.e., there are two positive numbers, k_1 and k_2, such that

$$k_1\left(\mathrm{d}x_1^2+\mathrm{d}x_2^2\right)<\langle \boldsymbol{M}\,\mathrm{d}x,\mathrm{d}x\rangle<k_2\left(\mathrm{d}x_1^2+\mathrm{d}x_2^2\right). \tag{2.8}$$

Therefore, β may be defined by Euclidean norm in the practice of stability analysis.

Theorem. *At every fixed total energy, a periodic trajectory in the system given by Eq.* (2.1) *is orbitally stable if and only if it is S-stable.*

Proof. See Appendix.

Remark 2. Synge's concept furnishes an analytical tool to study the orbital stability in strongly non-linear systems.

3. Stability analysis of normal modes

To analyze the stability of normal modes, the solution of the normal modes should be known beforehand. The solution