Available online at www.sciencedirect.com
SCIENCE DIRECT
International Journal of Non-Linear Mechanics 41 (2006) 665–671
INTERNATIONAL JOURNAL OF NON-LINEAR MECHANICS
www.elsevier.com/locate/nlm

A dual state variable model of the van der Pol oscillator

Alvin Post[a,*], Willem Stuiver[b]

[a]Department of Mechanical and Manufacturing Engineering Technology, Arizona State University—East, Mesa, AZ 85212, USA
[b]Department of Mechanical Engineering, University of Hawaii, Honolulu, HI 96822, USA

Received 1 October 2004; received in revised form 15 December 2005; accepted 13 January 2006

Abstract

A new, "dual" state variable (DSV) formulation is used to construct a model of the van der Pol oscillator. The model is valid for small degrees of non-linearity, and results are superior to those from a common perturbation technique, especially as non-linearity begins to increase. The DSV formulation utilizes a unique state space, and behavior in this space is illustrated for a wider range of non-linearity.

Keywords: van der Pol; Perturbation; DSV; Dual state variables; Oscillator

1. Introduction

The dual state variable (DSV) formulation provides a unique and useful structure for modeling non-linear oscillators. It has previously been applied to the non-linear pendulum and the Duffing oscillator in [1], and a description of its mathematical consequences is given there. In the DSV formulation, a second-order ordinary differential equation (ODE) of interest is paired with a "dual" ODE of the analyst's choosing, resulting in a formulation with four state variables. The two state variables representing the positions of the primary and dual oscillators are plotted against each other in a two-dimensional projection of the state trajectory. A separate projection of the state trajectory contains information about the rates of change of these variables.

When the dual equation and its initial conditions are judiciously chosen, state behavior can often exhibit a high degree of symmetry. That symmetry, along with some useful mathematical features of the formulation, can allow construction of accurate approximate solutions, even when non-linearities are large [1]. In this paper, the DSV formulation is applied to the van der Pol oscillator to achieve results that are superior to a perturbation technique. The analysis differs significantly from that described in [1] because the van der Pol oscillator has an asymmetric waveform, and it is non-conservative.

2. Summary of the dual state variable equation set

If the equation of interest, or primary equation, is stated as

$$\ddot{x} = F_1(x, \dot{x}, y, \dot{y}, t) \tag{1}$$

and the dual equation is stated as

$$\ddot{y} = F_2(y, \dot{y}, x, \dot{x}, t), \tag{2}$$

where t is the independent variable, then the (dual) state variable equations are defined as follows [1]:

$$\dot{x} = \alpha x - \beta y, \tag{3}$$

$$\dot{y} = \beta x + \alpha y, \tag{4}$$

$$\dot{\alpha} = \frac{xF_1 + yF_2}{x^2 + y^2} + \beta^2 - \alpha^2, \tag{5}$$

$$\dot{\beta} = \frac{xF_2 - yF_1}{x^2 + y^2} - 2\alpha\beta. \tag{6}$$

It is useful to employ polar coordinates ρ, ϕ when describing the state variables x and y. In particular, after conversion to polar coordinates, (3) and (4) become

$$\dot{\rho} = \alpha\rho \tag{7}$$

* Corresponding author. Tel.: +1 480 830 5864.
E-mail address: alvin.post@cox.net (A. Post).

0020-7462/$ - see front matter
doi:10.1016/j.ijnonlinmec.2006.01.004

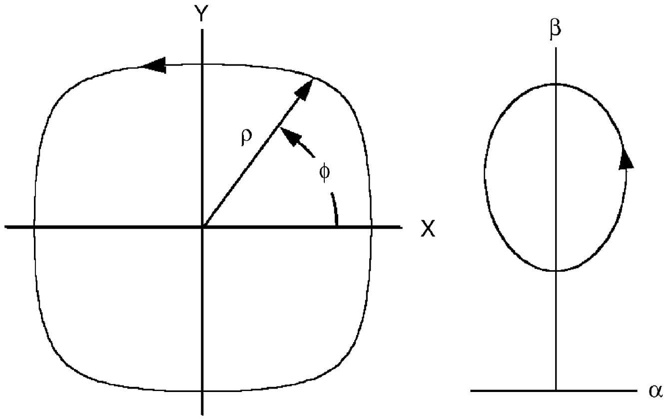

Fig. 1. A typical DSV state space representation of motion, in this case for the non-linear pendulum.

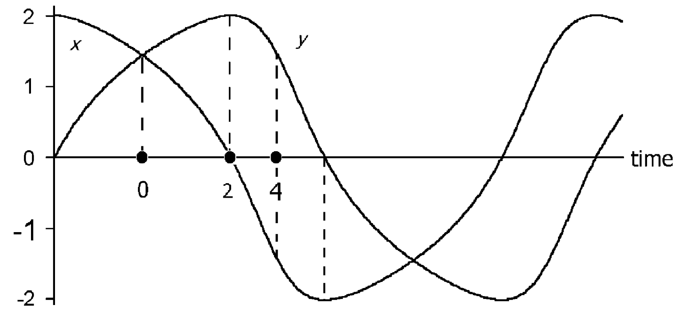

Fig. 2. A sketch of a van der Pol wave form x and that of a suitably chosen dual equation y. Selected reference points are noted.

and

$$\dot{\phi} = \beta. \tag{8}$$

The significance of α and β is apparent from these equations: α is a measure of the normalized rate of change of ρ, and β quantifies the angular motion of the x, y state point. The trajectory in the α, β projection of the state space therefore presents velocity information, and the α, β projection is consistent with the s-plane sometimes used in engineering analysis [1]. It is convenient to plot x, y in a two-dimensional projection of the state space, and α, β in a separate projection, as shown in Fig. 1.

Further analysis of the DSV equations shows that the solutions for $x(t)$ and $y(t)$ can be written as follows [1]:

$$x = e^{\int \alpha\, dt}\left[x_0 \cos\left(\int \beta\, dt\right) - y_0 \sin\left(\int \beta\, dt\right)\right], \tag{9}$$

$$y = e^{\int \alpha\, dt}\left[y_0 \cos\left(\int \beta\, dt\right) + x_0 \sin\left(\int \beta\, dt\right)\right]. \tag{10}$$

The functions α and β cannot generally be identified in closed form for non-linear ODEs, but approximations for them can often be devised based on a simplified model of state space behavior, and then used in (9) and (10) to approximate the motion of the oscillator.

3. The van der Pol oscillator and a "dual" equation

This oscillator has been studied in great detail for over 75 years [2]. The van der Pol equation is

$$\ddot{x} = -\mu\left(x^2 - 1\right)\dot{x} - ax, \tag{11}$$

where μ and a are positive constants. By the transformation $t \to \sqrt{a}t$, Eq. (11) can be recast as

$$\ddot{x} = -\lambda\left(x^2 - 1\right)\dot{x} - x, \tag{12}$$

where $\lambda = \mu/\sqrt{a}$ [2]. A periodic solution is known to exist for this oscillator, and the shape of the waveform has been qualitatively established [2]. This is the starting point for the analysis.

Any ODE of form (2) can serve as a dual equation, but the DSV approach is most useful when the dual equation in y is chosen to enhance symmetry in the x, y state space. Inspection of the van der Pol waveform $x(t)$ suggests that a vertically reflected and time-shifted version of the waveform, as shown in Fig. 2, will yield fairly symmetric behavior when y is plotted against x. In this figure, initial conditions are chosen so that peak values of y occur when x crosses the zero axis. Other choices can also lead to a certain degree of symmetry, for instance by allowing the peaks of the x and y oscillations to coincide, and such choices may be productive as well.

The waveform of the dual equation is seen to be a time-reversed version of the van der Pol equation. It can be constructed by transforming the van der Pol equation such that $x \to y$ and $t \to -t$. The resulting dual equation is

$$\ddot{y} = +\lambda\left(y^2 - 1\right)\dot{y} - y. \tag{13}$$

Note that (12) tends toward an asymptotically stable limit cycle, but for (13), the limit cycle is unstable.

4. Construction of the DSV state space plots

Use of the DSV formulation requires a thorough conceptual understanding of its unique state space. Because it is unfamiliar, an exposition of state space construction precedes mathematical analysis.

By reference to Fig. 2, a plot of x vs. y can be sketched (Fig. 3). The exact shape of the orbit cannot be inferred from Fig. 2, but symmetry about the 45° and 135° axes is clear. Rapidly and slowly evolving regions of the plot can be identified by comparing the rates of change at points 0 and 4 in Fig. 2.

A sketch of the α, β projection of the state trajectory is provided in Fig. 4. Construction of this graph begins with identification of the points at which the trajectory crosses the β-axis ($\alpha = 0$). By (7), this will occur whenever ρ is a maximum or a minimum, and by symmetry in the x, y plot, that occurs when the x, y orbit crosses the axes of symmetry at $\phi = 45°$ and 135°, i.e. when $y = \pm x$. These are points 0 and 4. Point 0 is associated with a slowly evolving point on the x, y orbit,