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# On the contact of a spherical membrane enclosing a fluid with rigid parallel planes

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ABSTRACT

The mechanical response of an inflated spherical membrane–fluid structure in contact with rigid parallel planes is studied. The membrane is assumed to be a two-dimensional non-linear elastic and isotropic structure, while no assumption is imposed on the fluid. A numerical procedure is employed to compute the equilibrium configurations of the membrane–fluid structure. This study provides information regarding the contact force, stress distribution and pressure in the membrane and in the enclosed fluid, respectively. It was observed that a transition between unwrinkled to partially wrinkled configurations of the membrane occurs subjected to the loading conditions. Further investigation of the wrinkled configurations is presented.



## 1. Introduction

In the present work the contact problem of a spherical membrane–fluid structure with rigid parallel planes is analyzed. The contact problem is formulated in terms of a general elastic isotropic strain energy function, a general fluid and large deformations. Examples of such membrane–fluid structures undergoing large deformation due to contact appear in many fields, e.g., biology (biological cells) and in the pharmaceutical industry (microcapsules).

In this work the membrane is considered to be a structure with thickness much smaller than any other dimension of the structure. It is common to model such a thin structure as merely a deformable surface and the thickness is not modeled explicitly but, rather, accounted for by the constitutive law. It is further assumed that the membrane stresses are only in-plane and bending stiffness vanishes. These assumptions are consistent with the leading-order model for small thickness of the three-dimensional theory and thus valid only for thin films.

This problem was first studied in [1] using the Mooney strain-energy function for the membrane. In [2] a similar contact problem was used to determine the mechanical properties of a cell membrane of an approximately spherical shape. In a more recent work [3], the constitutive law of an HSA-Alginate Capsules is determined by comparison of experimental data with a theoretical analysis based on [2]. In all the previous works mentioned above the presence of compressive stress is permitted which is not physically admissible in membrane structures. It is well known that for the case of shells, compressive stress may result in lost of stability (buckling) once the compressive stress is larger than some threshold related to the bending stiffness of the shell. For the case of shells with vanishing bending stiffness, namely membranes, the existence of compressive stress immediately gives rise to lost of stability in the form of wrinkling. In this study compressive stress are excluded, and membrane wrinkling is considered, which is a more realistic model for the membrane–fluid structure of interest. The conditions under which wrinkling appears and the domain of wrinkling are analyzed.

The wrinkling of thin-films has been studied by many researchers. The relaxed energy approach was introduced in [4] where the original energy was replaced by a relaxed energy that can accommodate wrinkling in the thin-film. This approach was used to study the mechanics of networks in [5], and in [6] an ideal fabric model which admits wrinkling was developed. Other works on wrinkling are [7] where a pulled spherical membrane is considered and in [8] an approximation to the amplitude of the wrinkling was developed.

In Section 2 the large deformation of the membrane is presented by the two principal stretches and the principal directions defined on the tangent plane of the membrane. The equilibrium equation in the membrane and the fluid are discussed in Section 3. Here also, the boundary conditions and the constitutive laws of the membrane and fluid are introduced. Section 4 describes in detail the numerical procedure employed, while the numerical results of a particular choice of membrane and fluid are presented in Section 5. The appearance of wrinkling

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in the membrane is studied in Section 6 and discussion of the results and conclusions are presented in Section 7.

## 2. Deformation

Consider a membrane $\Omega$ which in its reference configuration $\kappa$ occupied a spherical surface of radius $R$, and the reference configuration is taken to be a natural configuration. Define $\mathbf{X}$ to be the position of a material point $X$ on the reference sphere $\kappa$ such that

$$\mathbf{X} = R\mathbf{E}_R(\phi,\theta), \tag{1}$$

where the convected coordinates $\{\phi,\theta,R\}$ are the standard spherical coordinate inducing the orthonormal right-handed spherical basis $\{\mathbf{E}_\phi(\phi,\theta),\mathbf{E}_\theta(\theta),\mathbf{E}_R(\phi,\theta)\}$ with $\phi\in[0,\pi)$, $\theta\in[0,2\pi)$ and $R=const$. When an isotropic sphere comes to contact with two parallel rigid planes the deformation is necessarily axisymmetric (see Fig. 1). Hence, the position $\mathbf{x}$ of a material point $X$ on the close axisymmetric surface $\varphi$ can be represented by

$$\mathbf{x}(\phi,\theta) = r(\phi)\mathbf{e}_r(\psi,\theta), \tag{2}$$

where $\psi(\phi)$, and a second set of orthonormal right-handed spherical basis $\{\mathbf{e}_\psi(\psi,\theta),\mathbf{e}_\theta(\theta),\mathbf{e}_r(\psi,\theta)\}$ is introduced. The motion from the reference sphere $\kappa$ to the close axisymmetric deformed surface $\varphi$ is described by the map

$$\mathbf{x} = \chi_\kappa(\mathbf{X},t). \tag{3}$$

The deformation gradient

$$\mathbf{F} = \nabla\mathbf{x} = \mathbf{g}_1\otimes\mathbf{G}^1 + \mathbf{g}_2\otimes\mathbf{G}^2 \tag{4}$$

is expressed by the covariant basis $\mathbf{g}_i$ on $\varphi$ and the contravariant basis $\mathbf{G}^i$ on $\kappa$, where Latin indices have the range $(i=1,2)$. The covariant basis on $\varphi$ are defined by

$$\mathbf{g}_1 = \frac{\partial\mathbf{x}}{\partial\phi}, \quad \mathbf{g}_2 = \frac{\partial\mathbf{x}}{\partial\theta}, \tag{5}$$

and the covariant basis on $\kappa$ are

$$\mathbf{G}_1 = \frac{\partial\mathbf{X}}{\partial\phi}, \quad \mathbf{G}_2 = \frac{\partial\mathbf{X}}{\partial\theta}. \tag{6}$$

The contravariant basis on $\kappa$ satisfy the property

$$\mathbf{G}_i\cdot\mathbf{G}^j = \delta_i^j, \tag{7}$$

where $\delta_i^j$ is the Kronecker Delta. Calculation yields

$$\mathbf{g}_1 = r'\mathbf{e}_r + r\psi'\mathbf{e}_\psi, \quad \mathbf{g}_2 = r\sin\psi\,\mathbf{e}_\theta \tag{8}$$

and

$$\mathbf{G}_1 = R\mathbf{E}_\phi, \quad \mathbf{G}_2 = R\sin\phi\,\mathbf{E}_\theta, \tag{9}$$

where $()' = d()/d\phi$. By (7) the contravariant basis on $\kappa$ are

$$\mathbf{G}^1 = \frac{\mathbf{E}_\phi}{R}, \quad \mathbf{G}^2 = \frac{\mathbf{E}_\theta}{R\sin\phi}. \tag{10}$$

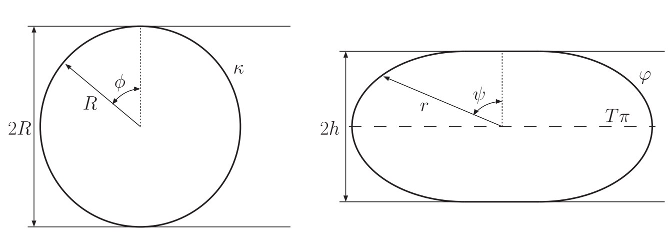


**Fig. 1.** Reference (left figure) and deformed (right figure) configurations of the membrane–fluid structure.

Now, using (8) and (10) the deformation gradient (4) takes the form

$$\mathbf{F} = \frac{r'\mathbf{e}_r + r\psi'\mathbf{e}_\psi}{R}\otimes\mathbf{E}_\phi + \frac{r\sin\psi}{R\sin\phi}\mathbf{e}_\theta\otimes\mathbf{E}_\theta, \tag{11}$$

which can be also expressed (see [9]) by the convenient form

$$\mathbf{F} = \lambda\mathbf{l}\otimes\mathbf{L} + \mu\mathbf{m}\otimes\mathbf{M}. \tag{12}$$

In (12) the coefficients $\lambda$ and $\mu$ are the principal stretches, the orthonormal vectors $\mathbf{l}$ and $\mathbf{m}$ belong to the tangent plane of $\varphi$ denoted by $T\varphi$ and the orthonormal vectors $\mathbf{L}$ and $\mathbf{M}$ belong to the tangent plane of $\kappa$ denoted by $T\kappa$. Comparison of (11) and (12) yields the relationships

$$\lambda = \frac{\sqrt{(r')^2+(r\psi')^2}}{R}, \quad \mu = \frac{r\sin\psi}{R\sin\phi},$$
$$\mathbf{l} = \frac{r'\mathbf{e}_r + r\psi'\mathbf{e}_\psi}{\lambda R}, \quad \mathbf{m} = \mathbf{e}_\theta, \quad \mathbf{L} = \mathbf{E}_\phi, \quad \mathbf{M} = \mathbf{E}_\theta, \tag{13}$$

and the areal dilation is

$$J = \sqrt{\det\mathbf{F}^T\mathbf{F}} = \lambda\mu. \tag{14}$$

Next, the unit vectors

$$\mathbf{N} = \mathbf{L}\times\mathbf{M}, \quad \mathbf{n} = \mathbf{l}\times\mathbf{m} \tag{15}$$

are the outward normals to $T\kappa$ and $T\varphi$, respectively.

The tangent unit vector $\mathbf{l}$ can also be expressed by the standard cylindrical basis $\{\mathbf{i}(\theta),\mathbf{j}(\theta),\mathbf{k}\}$ such that

$$\mathbf{l} = \cos\tau\,\mathbf{i} - \sin\tau\,\mathbf{k}, \tag{16}$$

when using the angle $\tau(\phi)$ and the unit normal to the parallel rigid planes $\mathbf{k}$, see Fig. 2. The transformations between the spherical and the cylindrical bases is given by

$$\mathbf{i} = \cos\psi\,\mathbf{e}_\psi + \sin\psi\,\mathbf{e}_r, \quad \mathbf{j} = \mathbf{e}_\theta, \quad \mathbf{k} = -\sin\psi\,\mathbf{e}_\psi + \cos\psi\,\mathbf{e}_r. \tag{17}$$

From $(13)_3$, (16) and (17) it can be shown that

$$r' = \lambda R\sin(\psi-\tau), \quad \psi' = \frac{\lambda R}{r}\cos(\psi-\tau), \tag{18}$$

which are first order ordinary differential equations for the functions $r(\phi)$ and $\psi(\phi)$.

## 3. Equilibrium

The referential equilibrium statement of the membrane is

$$\text{Div}\,\mathbf{P} + J\mathbf{f} = \mathbf{0}, \tag{19}$$

where Div is the referential two-dimensional divergence operator on $\kappa$, $\mathbf{P}$ is the first Piola–Kirchhoff stress tensor and $\mathbf{f}$ is the distributed force (body force and lateral traction) per unit area of $\varphi$.

Assuming the existence of a strain energy function $W(\mathbf{F})$ per unit area of $\kappa$ such that $\mathbf{P}$ is given by the gradient of the strain energy density with respect to $\mathbf{F}$ such that

$$\mathbf{P} = W_{\mathbf{F}}. \tag{20}$$

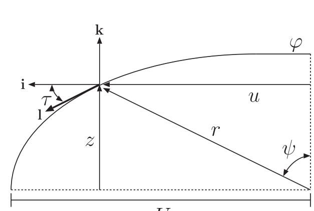


**Fig. 2.** The geometry of the deformed membrane and the tangent plane.