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Non-linear stresses in a rubber cylinder sheared by pressure at one end

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ABSTRACT

Normal stresses are set up by shearing a rubber block or tube. They depend strongly on the end conditions, even for relatively long specimens [A.N. Gent, J.B. Suh, S.G. Kelly III, Mechanics of rubber shear springs, Int. J. Non-Linear Mech. 42 (2007) 241–249; J.B. Suh, A.N. Gent, S.G. Kelly III, Shear of rubber tube springs, Int. J. Non-Linear Mech. 42 (2007) 1116–1126]. We have now examined a solid rubber cylinder bonded within a rigid cylindrical tube and subjected to pressure at one end. In this case, the correct end conditions for a simple shear deformation are met, at least approximately. Theoretical analysis and finite element calculations show that inwardly directed second-order stresses are set up at the wall, in contrast to the outwardly directed stresses generated by shearing a block or tube. However, for the particular geometry considered, the stresses were rather small in comparison with the applied pressure. Conditions are described under which they would be significantly larger. Stresses in a non-linearly viscous fluid under steady shear flows are expected to be similar, depending strongly on the geometry, end shapes and stress conditions.

1. Introduction

Previously, finite element calculations have been carried out for a rubber block bonded between two rigid parallel plates and sheared in plane strain by displacing one of the plates parallel to the other, Fig. 1 [1]. The rubber was assumed to be an incompressible elastic solid, obeying the simple neo-Hookean non-linear constitutive relation, with a single elastic coefficient, the shear modulus μ. For this geometry, Rivlin [2] showed that second-order normal stresses t_{11}, t_{22} and t_{33}, are required to maintain the shear deformation, in addition to the shear stress t_{12}:

$$t_{11} = \mu\gamma^2 + p \tag{1}$$

and

$$t_{22} = t_{33} = p \tag{2}$$

where p is a reference pressure. If the end pressure $t_{11} = 0$, then $t_{22} = t_{33} = -\mu\gamma^2$. Thus, the normal stress t_{22} is a second-order compressive stress.

Rivlin pointed out that other stresses must also be applied to the end surfaces to maintain this deformation. They are a stress t_n normal to the end surface in the deformed state, and a shear stress t_s (Fig. 1):

$$t_n = p - \mu\gamma^2/(1+\gamma^2) \tag{3}$$

$$t_s = \mu\gamma/(1+\gamma^2) \tag{4}$$

However, these stresses are not usually applied. Finite element calculations showed that strikingly different normal stresses are generated when they are absent [1]. The stress t_{22} becomes tensile instead of compressive, and the stress t_{11} parallel to the shear direction becomes positive and large, increasing in proportion to γ^2. These unexpected results showed that end conditions could not be neglected, even for long blocks.

Similar effects were found in shearing a cylindrical rubber tube bonded between inner and outer rigid cylindrical surfaces [3,4]. When the tube surface was displaced axially, then failure to apply the requisite stresses on the end surfaces led to high longitudinal stresses, as for sheared rectangular blocks, and the radial stresses became tensile instead of compressive.

We now consider a solid rubber cylinder bonded inside a rigid cylindrical tube and sheared by applying a pressure P to one end. The expected stresses are derived for a neo-Hookean material and compared with those found in one particular case by finite element analysis.

2. Theory

The rubber cylinder, of length L and radius a, is assumed to undergo a simple shear deformation with displacements only in the

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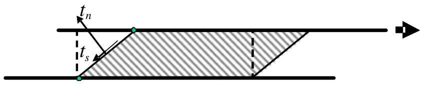

Fig. 1. Shear of a rubber block showing the stresses needed on the end surfaces to maintain a simple shear deformation [2].

length direction, the radial and azimuthal coordinates being unchanged. It is assumed to be incompressible in bulk and neo-Hookean in elastic behavior, with shear modulus μ. Using cylindrical coordinates r, θ, z, the stresses acting on a cylindrical element of the tube having a radial thickness dr are [2]:

$$t_{rz} = \mu\gamma$$

$$t_{zz} = \mu\gamma^2 + p$$

$$t_{rr} = t_{\theta\theta} = p$$

$$t_{r\theta} = t_{z\theta} = 0,$$

where γ is the amount of shear and p is an undefined pressure. The equations of equilibrium are:

$$\partial t_{rr}/\partial r + \partial t_{rz}/\partial z = 0 \qquad (5)$$

$$\partial t_{rz}/\partial r + \partial t_{zz}/\partial z + t_{rz}/r = 0 \qquad (6)$$

$$(1/r)\partial t_{\theta\theta}/\partial\theta = 0 \qquad (7)$$

Because γ is assumed to be independent of z, $\partial t_{rz}/\partial z$ is zero, and hence from Eq. (5), $\partial t_{rr}/\partial r$ and $\partial p/\partial r$ are also zero. Thus, p, t_{rr} and $t_{\theta\theta}$ are independent of r. By symmetry, p is also independent of θ. Eq. (6) then becomes

$$\mu\partial\gamma/\partial r + \partial p/\partial z + \mu\gamma/r = 0.$$

If we assume that $\partial p/\partial z$ is constant, this equation yields the small-strain solution for the dependence of the amount of shear on the radial distance r: $\gamma_r = (r/2\mu)(\partial p/\partial z)$. The maximum amount of shear at $r = a$ is thus

$$\gamma_m = (a/2\mu)(\partial p/\partial z).$$

The end conditions are:

$$\int 2\pi r t_{zz}\, dr = -\pi a^2 P, \quad \text{when } z = 0,$$

and

$$\int 2\pi r t_{zz}\, dr = 0, \quad \text{when } z = L.$$

Hence, $p(z = 0) = -P - \mu\gamma_m^2$ and

$$p(z = L) = -\mu\gamma_m^2$$

We infer that

$$p(z) = -P[1 - (z/L)] - \mu\gamma_m^2 \qquad (8)$$

The predicted wall stresses are

$$t_{rr} = t_{\theta\theta} = p(z)$$

$$t_{zz} = p(z) + \mu\gamma_m^2.$$

Thus, the second-order stresses are represented by the differences in the axial and radial normal stresses at any distance z along the wall:

$$t_{zz} - t_{rr} \ \text{or} \ t_{zz} - t_{\theta\theta} = \mu\gamma_m^2. \qquad (9)$$

3. Finite element analysis

The rubber cylinder was assigned a length L of 100 mm, radius a of 10 mm, and shear modulus μ of 1 MPa, and made incompressible in bulk. It was represented by 420 axisymmetric four-noded or eight-noded elements (CAX4H or CAX8H), distributed at higher density near the ends (see Fig. 2) in view of possible stress anomalies there. ABAQUS software was employed to determine stresses and displacements when a pressure P, ranging from 0.4 to 40 MPa, was applied to the upper end surface of the cylinder. Results are presented here only for the highest pressure, 40 MPa, and for eight-noded elements, the results for four-noded elements being indistinguishable.

The distribution of shear stress $t_{z\theta}$ at the wall is shown in Fig. 3. Over most of the cylinder length $t_{z\theta}$ is seen to be constant although there were marked departures near the ends, particularly at the pressurized end. The corresponding axial t_{zz} and radial t_{rr} normal stresses are shown in Figs. 4 and 5. They were approximately equal in

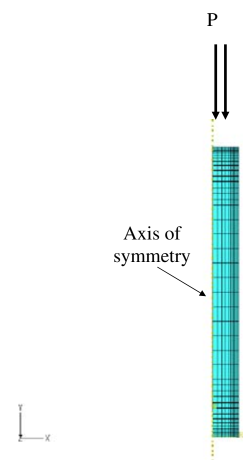

Fig. 2. FEA model of a pressurized cylinder.

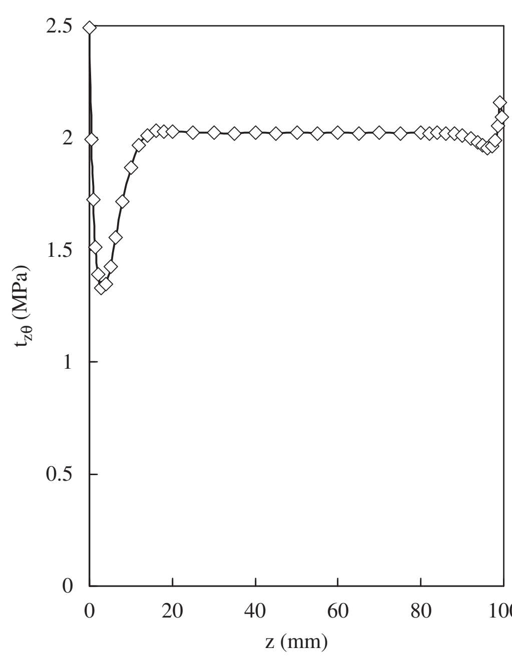

Fig. 3. Wall shear stress $t_{z\theta}$ vs distance z from the pressurized end. Applied pressure $P = 40$ MPa.