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Numerical simulation of interaction of solitary deformation waves in microstructured solids

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Abstract

In the present paper 1D wave propagation in microstructured solids is modelled based on the Mindlin theory and hierarchical approach. The governing equation under consideration is non-integrable therefore it is analysed numerically. Propagation and interaction of localised initial pulses is simulated numerically over long time intervals by employing the pseudospectral method. Special attention is paid to the solitonic character of the solution.

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1. Introduction and model equations

Wide application of microstructured materials (like alloys, crystallites, ceramics, functionally graded materials, etc.) in technology needs also proper testing methods in order to evaluate the properties of such materials. This need is especially acute because microstructural properties affect considerably the macrobehaviour of a compound material or a structure. In most general terms, microstructure means the existence of grains, inclusions, layers, block walls, etc., and the influence of anisotropy. There are powerful methods in continuum mechanics in order to describe the influence of such irregularities of media starting from early works of Cosserats and Voigt up to contemporary formulations. Corresponding models should be able to account for various scales of microstructure (see [1–4] and references therein). The scale-dependence involves dispersive as well as different non-linear effects and if they are balanced then solitary waves and/or solitons may emerge.

Solitary waves in microstructured solids are analysed using different models (see [4–6] and references therein). However, the crucial point related to the derivation of governing equations is to distinguish between non-linearities on macro- and microlevel together with proper modelling of dispersive effects. In [7–9] the Mindlin model [10] and hierarchical approach by Engelbrecht and Pastrone [4] is used in order to derive governing equations. By Mindlin [10], microstructured material is interpreted as an elastic continuum including microstructure that could be "a molecule of a polymer, a crystallite of a polycrystal or a grain of a granular material". This microstructure is modelled by microelements within the macrostructure. According to Eringen and Mindlin [1,10] fundamental balance laws should be formulated for macro- and microlevel separately. For 1D model this approach results in equations of motion in the following form:

$$\rho u_{tt} = \sigma_x,$$
$$I\psi_{tt} = \eta_x - \tau. \tag{1}$$

Here u is the macrodisplacement, ψ the microdeformation, ρ the macrodensity, I the microinertia, σ the macrostress, η the

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microstress and τ the interactive force. The free energy function is considered in the following form:

$$W = W_2 + W_3,$$
$$W_2 = \tfrac{1}{2}au_x^2 + \tfrac{1}{2}B\psi^2 + \tfrac{1}{2}C\psi_x^2 + D\psi u_x,$$
$$W_3 = \tfrac{1}{6}Nu_x^3 + \tfrac{1}{6}M\psi_x^3, \tag{2}$$

where a, B, C, D, M, N are constants. Here the quadratic term W_2 gives rise to the linear stress and the cubic W_3—to the non-linear part of stress. Then using the formulae

$$\sigma = \frac{\partial W}{\partial u_x}, \quad \eta = \frac{\partial W}{\partial \psi_x}, \quad \tau = \frac{\partial W}{\partial \psi} \tag{3}$$

Eqs. (1) are expressed in terms of variables u and ψ

$$\rho u_{tt} = au_{xx} + Nu_x u_{xx} + D\psi_x,$$
$$I\psi_{tt} = C\psi_{xx} + M\psi_x\psi_{xx} - Du_x - B\psi. \tag{4}$$

Next, slaving principle [4,7] is applied (in order to eliminate the microdeformation ψ from latter equations) and in terms of dimensionless variables $X = x/L$, $T = tc_0/L$, $U = u/U_0$, scale parameter $\delta = l^2/L^2$ (L and U_0 are amplitude and wavelength of the initial excitation, respectively; $c_0^2 = a/\rho$ and l is the scale of the microstructure) Eqs. (4) result in the hierarchical model equation

$$L_1 - \delta L_2 = 0,$$
$$L_1 = U_{TT} - bU_{XX} - \frac{\mu}{2}(U_X^2)_X,$$
$$L_2 = \left(\beta U_{TT} - \gamma U_{XX} - \delta^{1/2}\frac{\lambda}{2}U_{XX}^2\right)_{XX}, \tag{5}$$

where L_1 is macrostructure wave operator and L_2 microstructure wave operator. New dimensionless material constants b, μ, β, γ and λ are introduced during change of variables and they are directly related to constants a, B, C, D, M, N in free energy expression (2) (see [8,9] for details). If the scale parameter δ is small then the wave process is governed by properties of the macrostructure and vice versa, if δ is large, then properties of the microstructure govern the process.

For future analysis Eq. (5) is expressed in terms of deformation $v = U_X$ and lower-case letters x and t are used for dimensionless coordinate and time.

$$v_{tt} - bv_{xx} - \frac{\mu}{2}(v^2)_{xx} - \delta(\beta v_{tt} - \gamma v_{xx})_{xx} + \delta^{3/2}\frac{\lambda}{2}[(v_x)^2]_{xxx} = 0. \tag{6}$$

The full derivation of governing equation (6) can be found in [7,8].

Eq. (6) is non-integrable but it is possible to find its travelling wave solution $v(x - ct)$ in the form of an asymmetric solitary wave using numerical integration under asymptotic boundary conditions (i.e. $u, u_x, u_{xx}, \ldots \to 0$, if $x \to \pm\infty$). The analytic conditions for the existence of solitary waves modelled by Eq. (6) are given by Janno and Engelbrecht in [8,9]:

$$\frac{c^2 - b}{\beta c^2 - \gamma} > 0, \quad \left(\frac{\beta c^2 - \gamma}{c^2 - b}\right)^3 > \frac{4\lambda^2}{\mu^2},$$
$$\mu \neq 0, \quad \beta c^2 - \gamma \neq 0, \quad c^2 - b \neq 0. \tag{7}$$

In the case of $\lambda = 0$ the non-linearity in the microscale is neglected and Eq. (6) admits bell-like solitary wave solution [6,9]

$$v(x - ct) = A\,\mathrm{sech}^2\frac{\varkappa(x - ct)}{2},$$
$$A = \frac{3(c^2 - b)}{\mu}, \quad \varkappa = \sqrt{\frac{c^2 - b}{\delta(\beta c^2 - \gamma)}}. \tag{8}$$

From the viewpoint of soliton dynamics, three problems are of importance: the existence of solitary waves, the emergence of solitary waves and the interaction of solitary waves. The latter is important in order to prove the solitonic character of solitary waves, i.e. to understand whether solitary waves are able to propagate at constant speed and shape and to restore these quantities after interactions. If yes, these solitary waves are called solitons. Here in this paper the basic model is a two-wave equation with complicated dispersive and non-linear terms. The existence of solitary waves is proved by Janno and Engelbrecht [8,9], the preliminary analysis of emergence of trains of solitary waves is presented in our earlier study [11] and here we present the preliminary results on interaction of solitary waves. The notion of solitary waves is used because the elastic interaction should prove whether these waves are solitons or not. As it is shown below, the problem is complicated and needs further analysis.

2. Statement of the problem and numerical technique

In the present paper the propagation and the interaction of localised initial pulses in microstructured materials (governed by Eq. (6)) is simulated numerically over long time intervals. Two goals are stated (i) to examine the solitonic character of the solution and (ii) to estimate the influence of the microlevel non-linear parameter λ on the solution.

For this reason Eq. (6) is integrated numerically under localised initial conditions

$$v(x, 0) = \sum_{i=1}^{2} A_i^0\,\mathrm{sech}^2\frac{\varkappa_i(x - \xi_i)}{2}, \quad 0 \leqslant x < 2k\pi. \tag{9}$$

Initial amplitudes A_i^0 and the widths $\varkappa_i$ ($i = 1, 2$) correspond to different initial speeds $c_1 \neq c_2$, ξ_i are initial phase shifts and k is integer. It is clear that in case $c_1c_2 < 0$ head-on collision and in case of $c_1c_2 > 0$ overtaking interaction takes place (if periodic boundary conditions are applied then this is true as in case $c_1 > c_2$ as well as in case $c_1 < c_2$).

For numerical integration discrete Fourier transform (DFT) based pseudospectral method (PsM) [12,13] is used and