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Review

A review of simple formulae for elastic hoop stresses in cylindrical and spherical pressure vessels: What can be used when

G.B. Sinclair*, J.E. Helms

Department of Mechanical Engineering, Louisiana State University, Baton Rouge, LA 70803, USA

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ABSTRACT

Classical simple formulae for elastic hoop stresses in cylindrical and spherical pressure vessels continue to be used in structural analysis today because they facilitate design procedures. Traditionally such formulae are only applied to thin-walled pressure vessels under internal pressure. There do exist, however, some variations of these formulae that remain simple yet permit wider use. Here, by reviewing various underlying rationales for simple hoop stress formulae, we make a determination of when and how well different formulae apply. For the formulae that do apply to thicker vessels than usually recognized, we give companion results for external pressure.

1. Introduction

Distinguished by the subscript c, the *classical formulae* for the elastic hoop stress, σ, produced by an internal gauge pressure p acting within thin-walled pressure vessels have

$$\sigma_c = \frac{pr}{t}, \quad \sigma_c = \frac{pr}{2t}, \tag{1}$$

for cylindrical and spherical vessels, respectively. In eqn (1), r and t are corresponding inner radii and thicknesses.

Possibly the earliest development of an expression like the first of eqn (1) is that obtained experimentally by Mariotte circa 1670, [1]. Mariotte tested closed thin-walled cylindrical tanks by connecting them to standpipes placed on top of them. He found that the height of the water in the standpipes when cylinders burst was proportional to the cylinder's wall thickness and inversely proportional to its radius. That is, in effect, $p \propto t/r$ at rupture. Thus Mariotte experimentally confirmed the essential elements of the first of eqn (1).

It is not clear to us when the explicit expressions for the hoop stresses in eqn (1) were developed. An alternative form for the first of eqn (1) was provided by Barlow circa 1830, [2]. Identified with the subscript B, this has

$$\sigma_B = \frac{pR}{t}, \tag{2}$$

for cylinders, where $R = r + t$ is the external radius. Otherwise we are unaware of just who developed eqn (1) and when. It would seem likely, however, that this occurred before Lamé's derivation of the more complex formulae for hoop stresses in thick-walled cylindrical and spherical vessels. This derivation was reported in Ref. [3] in 1852.

The classical formula for the hoop stress in cylinders, the first of eqn (1), has been given in strength of materials texts, and occasionally in statics texts, for over a century. Examples of the former are [4,5]. An example of the latter is [6]. For more than 50 years this formula has been given in introductory elasticity books [7,8], texts on shell theory [9], design books [10], and engineering handbooks [11].

This formula continues to be cited in modern mechanics of materials texts right up to the present time. In chronological order, examples are [12–19]. It is also given in current introductory solid mechanics books [20,21] and machine design books [22–25], as well as current/recent handbooks [26–28]. At this time it would appear to be fair to say that the classical formula for the hoop stress in cylinders under internal pressure has gained long-standing and widespread acceptance.

Similarly the classical formula for the hoop stress in spheres, the second of eqn (1), has been given in texts of yore [4,6,7,9,29]. It too continues to be given in current/recent texts including all of the

* Corresponding author. Tel.: +1 (225) 767 5786.
E-mail addresses: glennsinclair@cox.net, sinclair@me.lsu.edu (G.B. Sinclair), helms@me.lsu.edu (J.E. Helms).

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modern mechanics of materials texts previously cited, as well as an introductory solid mechanics text and some handbooks [20,26,27]. It is not usually given in machine design books. Nonetheless it would seem reasonable to view the classical formula for the hoop stress in spheres under internal pressure as having also gained long-standing and widespread acceptance.

The formulae of eqn (1) only apply away from any discontinuities that cause stress concentrations, and provided pressure vessel walls are sufficiently thin. The most commonly accepted range of sufficiently small wall thicknesses is

$$t/r \leq 1/10. \tag{3}$$

That is, that the thickness be at least one order of magnitude smaller than the inner radius: requirements of this nature are common in engineering when defining relatively small dimensions (see, e.g., [12]). On occasion, further justification for the range in inequality (3) is offered by noting that it leads to no more than about a 5% deviation from the maximum hoop stresses [13,18]. Though less frequent than inequality (3), there are some other upper limits for applying eqn (1) given in the literature. Examples are $t/r \leq 1/20$ in Ref. [24], $t/r \leq 1/5$ in Refs. [22,27]. When upper limits are exceeded, presumably formulae for thick-walled vessels are to be used.

These *Lamé formulae* for the maximum hoop stress, $\sigma_{\max}$, produced by internal pressure within thick-walled pressure vessels have

$$\sigma_{\max} = \frac{p\left(R^2 + r^2\right)}{t(R + r)}, \quad \sigma_{\max} = \frac{p\left(R^3 + 2r^3\right)}{2t\left(R^2 + Rr + r^2\right)}, \tag{4}$$

for cylindrical and spherical vessels, respectively. Derivations of the formulae in eqn (4) may be found in Ref. [30]: therein it is also explicitly pointed out that in the limit as $t \to 0$ the stresses in eqn (4) recover their counterparts in eqn (1). Clearly, therefore, there has to be some sufficiently small range of thicknesses such that eqn (1) suffice. Within this range, then, the relative simplicity of eqn (1) facilitates design by enabling the ready determination of design t given allowable σ and specified p and r. This simplicity is a key reason for the continued use of formulae like eqn (1) today. A detailed and precise explanation of the role of these formulae in the design of pressure vessels can be found in the ASME code [31], or the exposition of this code in Megyesy [32].

On the other hand, as $t \to \infty$ in eqn (4), $\sigma_{\max} \to p, p/2$, respectively, in marked contrast to eqn (1) as $t \to \infty$ which has $\sigma \to 0$ for both cylinders and spheres. Hence there has to be some upper limit or limits on t/r for eqn (1) to provide reasonable respective estimates of eqn (4). One objective of the present review is to check inequality (3) in this role and, if needed, to augment it so that upper limits are set in a clear and consistent way.

While eqn (1) contain by far the most common simple expressions for hoop stresses in the literature, there are also some variations in these expressions themselves. One such is the already noted expression of Barlow for cylinders, eqn (2). Another for cylinders under internal pressure has

$$\sigma_S = \frac{p\bar{r}}{t}, \tag{5}$$

where $\bar{r} = r + t/2$ is the mean radius. As far as we can discern, this formula was first given in Shigley's Mechanical Engineering Design circa 1972, thus the subscript S. It continues to be given in the current version of Shigley [23]. It is also provided in Refs. [24,32–34]. A further alternative for spheres under internal pressure has

$$\sigma_R = \frac{p\bar{r}}{2t}, \tag{6}$$

the analogue of eqn (5) in effect. As far as we can discern, this formula is only given in Roark's Formulas for Stress and Strain [28], thus the subscript R.

Two other simple formulae are furnished in the ASME code [31]. Distinguished by the subscript d for design code, these have

$$\sigma_d = \sigma_c + 0.6p, \sigma_d = \sigma_c + 0.1p, \tag{7}$$

for cylinders and spheres under internal pressure, respectively. Precise ranges of application for these formulae are set out in Ref. [31] ($t/r \leq 0.500$, 0.356, respectively).

While it seems certain that the various alternatives of eqns (5)–(7) would not have been put forward unless they offered some advantages over their counterparts in eqn (1), it is not obvious from any of the references listed here just exactly what these advantages are. Accordingly, as this review continues we seek to try and clarify the relative merits of eqns (5)–(7) versus eqn (1).

All of the foregoing applies to internal pressure. With external pressure, buckling is possible. Consequently most discussions of classical formulae in the literature preclude their use with external pressure because of a sense that, in the limited thickness ranges in which formulae like eqn (1) could be physically appropriate, buckling occurs. There are instances in the literature, however, which, recognizing that buckling may not always so dominate, suggest that then eqn (1) apply with a sign change. That is, the hoop stress with external pressure, σ_e, is simply given by

$$\sigma_e = -\sigma, \tag{8}$$

where σ is as in eqn (1) or eqn (5) for cylinders, eqn (1) or eqn (6) for spheres. This view of the hoop stress produced by external pressure on cylinders is stated in Refs. [15,28,35,36] for the first of eqn (1) and [32] for eqn (5). For spheres, it is stated in Refs. [35,36] for the second of eqn (1), and [28] for eqn (6). Here, for instances in which buckling does not preclude their use, we also seek to review the accuracy of eqn (1), eqn (5) and eqn (6) in conjunction with eqn (8) when pressure is applied externally.

In the remainder of this review, we begin (Sect. 2) with a recap of the traditional statics derivations underlying eqn (1), followed by systematic improvements afforded by synthetic division. Shigley's formula (5) is a natural outcome. The resulting formulae for hoop stresses are lower bounds. Thus next (Sect. 3) we develop upper bounds. Barlow's formula (2) is one outcome of this exercise. Thereafter (Sect. 4) we combine lower and upper bounds. This leads to design formulae, including those of ASME in eqn (7). Then (Sect. 5) we examine what formulae apply with external pressure. In light of this review, we conclude (Sect. 6) by summarizing what simple formulae can be applied when.

2. Lower bounds

A straightforward way to obtain lower bounds is to determine average hoop stress values using equilibrium: being averages, these hoop stresses have to be lower bounds for hoop stress maxima.

To this end, first we view the free-body diagram in Fig. 1 as being for half of a cross section of a *cylindrical pressure vessel* under internal pressure. Then balancing forces per unit out-of-plane length gives

$$\bar{\sigma} = \frac{pr}{t} = \sigma_c, \tag{9}$$

where $\bar{\sigma}$ is the average cylindrical hoop stress and σ_c is as in the first of eqn (1). Accordingly, as is well recognized, this σ_c is an average value and so a lower bound.