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A new branch of solutions of boundary-layer flows over a permeable stretching plate

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Abstract

The steady-state boundary-layer flows over a permeable stretching sheet are investigated by an analytic method for strongly non-linear problems, namely the homotopy analysis method (HAM). Two branches of solutions are obtained. One of them agrees well with the known numerical solutions. The other is new and has never been reported in general cases. The entrainment velocity of the new branch of solutions is always smaller than that of the known ones. For permeable stretching sheet with sufficiently large suction of mass flux, the difference between the shear stresses and velocity profiles of two branches of solutions is obvious: the shear stress of the new branch of solutions is considerably larger than that of the known ones. However, for impermeable sheet and permeable sheet with injection or small suction of mass flux, the shear stress and the velocity profile of two branches of solutions are rather close: in some cases the difference is so small that the new branch of solutions might be neglected even by numerical techniques. This reveals the reason why the new branch of solutions has not been reported. This work also illustrates that, for some non-linear problems having multiple solutions, analytic techniques are sometimes more effective than numerical methods.

Keywords: Similarity solution; Boundary-layer; Stretching wall; Permeable surface; Homotopy analysis method

1. Introduction

The investigation of boundary-layer flows of incompressible viscous fluid over a stretching sheet has many important applications in engineering, such as aerodynamic extrusion of plastic sheets, boundary layers along liquid film condensation process, cooling process of metallic plate in a cooling bath, glass and polymer industries, and process of hot rolling, wire drawing and paper production. The investigations were made by a lots of researchers, such as Sakiadis [1], Crane [2], Banks [3], Banks and Zaturska [4], Grubka and Bobba [5], Ali [6] for the impermeable plate and Erickson et al. [7], Gupta and Gupta [8], Chen and Char [9], Chaudhary et al. [10], Elbashbeshy [11], Pop and Na [12], Magyari and Keller [13], and Magyari et al. [14] for the permeable plate. McLeod and Rajagopal investigated the uniqueness of flows of Navier–Stokes fluid due to a stretching boundary [15]. The non-uniqueness of flows of non-Newtonian fluids over a stretching sheet was investigated by Chang et al. [16] and Lawrence et al. [17].

Consider boundary-layer flows over a stretching permeable sheet, governed by

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2}$$

subject to the boundary conditions

$$u=U_w=a\ (x+b)^{\lambda},\quad v=V_w=A(x+b)^{(\lambda-1)/2}\quad \text{at } y=0, \tag{3}$$

$$u\to 0\quad \text{as } y\to+\infty, \tag{4}$$

where $U_w=a(x+b)^{\lambda}$ denotes the stretching velocity of the sheet, $V_w=A(x+b)^{(\lambda-1)/2}$ the velocity of fluid through the permeable sheet, respectively. Note that $A>0$ corresponds to the injection, and $A<0$ the suction of mass flux, respectively.

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Following Crane [2] and Banks [3], we transfer the above non-linear partial differential equations into a set of ordinary differential equations. Let ψ denote the stream function. Under the similarity transformation:

$$\psi = a\sqrt{\frac{v}{a(1+\lambda)}}(x+b)^{(\lambda+1)/2}F(\xi),$$

$$\xi = \sqrt{\frac{a(1+\lambda)}{v}}(x+b)^{(\lambda-1)/2}y, \tag{5}$$

where $a \neq 0$ and $a(1+\lambda) > 0$, the original equations become

$$F'''(\xi) + \tfrac{1}{2}F(\xi)F''(\xi) - \beta F'^2(\xi) = 0, \tag{6}$$

subject to the boundary conditions

$$F(0) = \alpha, \quad F'(0) = 1, \quad F'(+\infty) = 0, \tag{7}$$

where

$$\alpha = -\frac{2A}{\sqrt{a(\lambda+1)v}}, \quad \beta = \frac{\lambda}{1+\lambda}. \tag{8}$$

Here, $\alpha=0$ denotes the impermeable sheet, $\alpha<0$ and $\alpha>0$ correspond to the lateral injection ($V_w>0$) and suction ($V_w<0$) of mass flux through the permeable sheet, respectively. Since $a(1+\lambda)>0$, it holds $a>0$ when $\lambda>-1$, corresponding to $-1<\beta\leqslant 1$; and $a<0$ when $\lambda<-1$, corresponding to $\beta>1$. When $a<0$, the stretching velocity of the sheet is negative, corresponding to the so-called backward boundary-layer flows that have physical meanings, as pointed out by Goldstein [18] and Kuiken [19].

From (5), we have the velocity field of the fluid

$$u(x,y) = a(x+b)^{\lambda}F'(\xi), \tag{9}$$

$$v(x,y) = -\tfrac{1}{2}\sqrt{a(1+\lambda)v}(x+b)^{(\lambda-1)/2} \times [F(\xi) + (2\beta-1)\xi F'(\xi)]. \tag{10}$$

The entrainment velocity of the fluid is given by

$$v(x,+\infty) = -\tfrac{1}{2}\sqrt{a(1+\lambda)v}(x+b)^{(\lambda-1)/2}F(+\infty), \tag{11}$$

and the shear stress on the stretching sheet is

$$\tau_w = -\rho v \left.\frac{\partial u}{\partial y}\right|_{y=0} = -a\rho\sqrt{a(1+\lambda)v}(x+b)^{(3\lambda-1)/2}F''(0). \tag{12}$$

Thus, $F''(0)$ and $F(+\infty)$ have clear physical meanings. For simplicity, we call $F(+\infty)$ the entrainment parameter.

The non-linear two-point boundary-value problem mentioned above is not easy to solve even by means of numerical techniques. Using the shooting method, Banks [3] gave a branch of numerical solutions when $-1<\beta<+\infty$ for impermeable plate ($\alpha=0$). Banks' solutions exist for $-1<\beta<+\infty$ and have the property $f'(\xi)>0$ in the whole region $0\leqslant\xi<+\infty$. Banks [3] showed that the boundary-layer flow problem does not admit similarity solutions for $\beta<-1$. As mentioned by Liao and Pop [20], when $\lambda>-1$, corresponding to a restricted region $-1<\beta\leqslant 1$, the boundary-layer flow over an impermeable plate ($\alpha=0$) is mathematically equivalent to (but physically different from) the steady free convection flow over a vertical semi-infinite plate embedded in a fluid-saturated porous medium of ambient temperature T_∞, governed by the so-called Cheng–Minkowycz's equation [21]. But, different from Banks [3], Ingham and Brown [21] found two branches of solutions in case of $\frac{1}{2}\leqslant\beta\leqslant 1$. Therefore, at least in case of $\frac{1}{2}<\beta\leqslant 1$, the boundary-layer flows over a stretching impermeable plate ($\alpha=0$) should have two branches of solutions. Currently, exact solutions for boundary-layer flows over a permeable stretching sheet at some special values of β, such as $\beta=-\frac{1}{2}$ and $\beta=\frac{1}{2}$, are given by Magyari and Keller [13]. When $\beta=\frac{1}{2}$ and $\beta=-\frac{1}{2}$, one has the close-form solution

$$F(\xi) = \frac{\alpha+\sqrt{\alpha^2+8}}{2}\left\{1 - \frac{(\alpha-\sqrt{\alpha^2+8})^2}{8} \times \exp\left[\left(-\frac{\alpha+\sqrt{\alpha^2+8}}{4}\right)\xi\right]\right\} \tag{13}$$

and

$$F(\xi) = \sqrt{\alpha^2+4}\tanh\left[\left(\frac{\sqrt{\alpha^2+4}}{4}\right)\xi + \mathrm{sign}(F_w)\mathrm{ArcSech}\left(\frac{2}{\sqrt{\alpha^2+4}}\right)\right], \tag{14}$$

respectively, where $\tanh(z)$ is the hyperbolic tangent of z, $\mathrm{ArcSech}(z)$ is the inverse hyperbolic secant of z, and $\mathrm{sign}(x)$ is defined by

$$\mathrm{sign}(x) = \begin{cases} 1, & x\geqslant 0, \\ -1, & x<0. \end{cases}$$

For details, refer to Magyari and Keller [13]. In a physically different but mathematically identical context, Chaudhary et al. [10] showed that the similarity solutions also exist for $\beta<-1$ if suction ($\alpha>0$) is applied at the surface. Currently, Magyari et al. [14] showed that, in the special case $\lambda=-1$, there exist multiple solutions for the permeable plate with sufficiently large suction. Liao and Magyari [22] found that there exist an infinite number of algebraically decaying solutions of the boundary-layer flows over a stretching impermeable sheet in case of $-\frac{1}{2}<\lambda<0$. All of these support the conclusion that there exist multiple solutions for the considered problem at hand. However, to the best of our knowledge, multiple solutions of the boundary-layer flows over a stretching permeable sheet ($\alpha\neq 0$) have never been reported in general cases of $\lambda\neq -1$.

Generally speaking, it is not easy to solve a non-linear differential equation. Based on the homotopy [23], a basic concept in topology [24], some powerful numerical techniques for non-linear problems, such as the continuation method [25] and the homotopy continuation method [26–30], were developed. There is a suite of FORTRAN subroutines in Netlib for numerically solving non-linear systems of equations by homotopy methods, called HOMPACK. Recently, a kind of analytic method, namely the homotopy analysis method (HAM)