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Shell analysis of thin-walled pipes. Part II – Finite element formulation

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A B S T R A C T

A finite element formulation is developed for the analysis of thin-walled pipes based on thin shell theory. The formulation starts with a Fourier series solution of the equilibrium equations developed in a companion paper and develops a family of exact shape functions for each mode. The shape functions developed are used in conjunction with the principle of stationary potential energy and yield a finite element that is exact within the assumptions of the underlying shell formulation. The stiffness matrix contribution for each mode n is observed to be fully uncoupled from those based on other modes $m \neq n$. The resulting finite element is shown to be free from discretization errors normally occurring in conventional finite elements. The applicability of the solution is illustrated through examples with various loading cases and boundary conditions. A comparison with other finite element and closed form solutions demonstrates the validity and accuracy of the current finite element.

1. Introduction and scope

In a companion paper [1], the governing equilibrium conditions and possible boundary conditions for thin-walled pipes were developed and solved using a Fourier series solution. A closed form solution was developed for two practical examples. While the solutions developed are exact within the limitations of the assumptions, the solution process was found to be particularly lengthy to conduct by hand. In this context, the present paper attempts to preserve the exactness of the solution while automating it by implementing a finite element solution based on the exact solution of the field equations already developed.

2. Literature review

A comprehensive review on finite element formulations for the analysis of pipes is provided in the series of review papers [2–5]. Most of the formulations were related to pipe bends subjected to specific load patterns. For the most part, the work is applicable to straight pipes (as straight pipes can be conceived as elbows with infinite radius of curvature). A comparative review of strain–displacement relationships in various shell formulations was presented in the companion paper [1].

In all the research discussed below, the kinematic assumptions of the Love–Kirchhoff thin shell theory are adopted. These are (a) straight fibres perpendicular to the middle surface before deformation remain straight and perpendicular to the middle surface after deformation, and (b) the radial normal stress and transverse shear stresses are neglected.

This includes the work of Ohtsubo and Watanabe [6] who developed a finite element as an assembly of ring elements. Their formulation captures shear deformation and warping effects but assumes inextensible hoop strains. The formulation is applicable to both in-plane and out of plane bending. The displacement fields were assumed to have a harmonic distribution in the circumferential direction. In the longitudinal direction, the displacement fields were interpolated using Hermitian polynomials.

Bathe and Almeida [7] developed a four node finite element for the linear analysis of elbows with large bend radii by assuming a cubic Lagrangian interpolation for the tangential and radial displacements over the length of the elbow. Their formulation neither captured warping deformations nor radial expansion. Subsequent improvements included developing interaction effects between elbow elements and straight pipe segments [8] and developing the non-linear capabilities for the element [9].

Militello and Huespe [10] further improved the element by Bathe and Almeida [7] by capturing warping deformations by interpolating the longitudinal displacements with cubic Lagrangian polynomials. They expanded the tangential and radial displacements using a limited number of Fourier series terms.

Another improvement to Bathe and Almeida's element was conducted by Abo-Elkhier [11] who adopted more complete displacement–strain relations in their formulation.

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The pipe inextensibility assumption in the radial direction was relaxed in the work of Yan et al. [12] who developed a formulation for plastic limit load of pipe elbows. The remainder of their formulation is consistent with that of Militello and Huespe [10]. The longitudinal, tangential and radial displacements were all expressed as Fourier series in the circumferential direction with cubic Lagrangian interpolation polynomials for the longitudinal displacement and Hermitian polynomials for the tangential and radial displacements.

Karamanos [13] broadened the previous work to model buckling instability of pipes. His work included non-linear elastic effects and investigated the relationship between ovalization and buckling of pipes subjected to constant bending moments. His work was followed up by Karamanos et al. [14] who included the effects of plasticity and internal pressure.

2.1. Expression for the potential energy

The total potential energy $\prod^*$ of the system for a pipe element of length ℓ, mid-surface radius r and thickness t, undergoing displacements u, v and w in the z (longitudinal), ϕ (tangential) and ρ (radial) directions, respectively, under externally applied tractions $q_u(z,\phi)$, $q_v(z,\phi)$, and $q_w(z,\phi)$ acting on the middle surface of the pipe [1] is

$$\prod^* = \frac{1}{2}\int_{z=0}^{\ell}\int_{\phi=0}^{2\pi}\Bigg\langle \frac{Et}{1-\nu^2}\Bigg\{\left(\frac{\partial u}{\partial z}\right)^2+\frac{1}{r^2}\left[w^2+2w\frac{\partial v}{\partial \phi}+\left(\frac{\partial v}{\partial \phi}\right)^2\right] + 2\nu\left(\frac{\partial u}{\partial z}\right)\left(\frac{1}{r}\right)\left(w+\frac{\partial v}{\partial \phi}\right)\Bigg\} + Gt\left[\left(\frac{\partial v}{\partial z}\right)^2+\frac{2}{r}\frac{\partial v}{\partial z}\frac{\partial u}{\partial \phi} + \frac{1}{r^2}\left(\frac{\partial u}{\partial \phi}\right)^2\right] + \frac{Et^3}{12(1-\nu^2)}\Bigg\{\left(\frac{\partial^2 w}{\partial z^2}\right)^2+\frac{1}{r^4}\left[\left(\frac{\partial^2 w}{\partial \phi^2}\right)^2 - 2\left(\frac{\partial^2 w}{\partial \phi^2}\right)\left(\frac{\partial v}{\partial \phi}\right)+\left(\frac{\partial v}{\partial \phi}\right)^2\right] + \frac{2\nu}{r^2}\left(\frac{\partial^2 w}{\partial z^2}\right)\left(\frac{\partial^2 w}{\partial \phi^2}-\frac{\partial v}{\partial \phi}\right)\Bigg\} + \frac{Gt^3}{3r^2}\left[\left(\frac{\partial^2 w}{\partial z\partial \phi}\right)^2 - 2\left(\frac{\partial^2 w}{\partial z\partial \phi}\right)\left(\frac{\partial v}{\partial z}\right)+\left(\frac{\partial v}{\partial z}\right)^2\right] - [q_u(z,\phi)u+q_v(z,\phi)v+q_w(z,\phi)w]\Bigg\rangle r\mathrm{d}\phi\mathrm{d}z \quad (1)$$

In Eq. (1), ν is Poisson's ratio, E is Young's Modulus of the pipe material and G is the shear modulus of the pipe material.

2.2. Displacement fields

Consistently with the formulation in the companion paper [1], the mid-surface displacements are expressed as Fourier expansions of the coordinate ϕ as:

$$\begin{aligned} u(z,\phi) &= a_0(z)+\sum_{n=1}^{\alpha}a_n(z)\cos n\phi+\sum_{n=1}^{\alpha}b_n(z)\sin n\phi \\ v(z,\phi) &= c_0(z)+\sum_{n=1}^{\alpha}c_n(z)\cos n\phi+\sum_{n=1}^{\alpha}d_n(z)\sin n\phi \\ w(z,\phi) &= f_0(z)+\sum_{n=1}^{\alpha}f_n(z)\cos n\phi+\sum_{n=1}^{\alpha}g_n(z)\sin n\phi \end{aligned} \quad (2a-c)$$

In Eqs. (2a–c), functions $a_0(z)$, $a_n(z)$, $b_n(z)$, $c_0(z)$, $c_n(z)$, $d_n(z)$, $f_0(z)$, $f_n(z)$ and $g_n(z)$ are displacement functions to be determined from equilibrium considerations. When α is infinite, the series solution converges to the exact solution of the problem within the limitations of the assumptions made. In practicality, the series solution will be truncated as α is chosen as a finite number.

2.3. Formulation of exact shape functions

The closed form solution for the unknown displacement functions $a_0(z),\ldots,g_n(z)$, which satisfy the homogeneous part of the equilibrium conditions, were derived in the companion paper [1] as:

$$\begin{aligned} a_0(z) &= \langle e_0(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{A}_0\}_{8\times 1}, \quad c_0(z) = \langle \overline{\overline{C}}_0(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{A}_0\}_{8\times 1}, \\ f_0(z) &= \langle \overline{\overline{F}}_0(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{A}_0\}_{8\times 1} \\ a_n(z) &= \langle \overline{\overline{A}}_n(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{F}_n\}_{8\times 1}, \quad d_n(z) = \langle \overline{\overline{D}}_n(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{F}_n\}_{8\times 1}, \\ f_n(z) &= \langle e_n(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{F}_n\}_{8\times 1} \\ b_n(z) &= \langle \overline{\overline{A}}_n(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{G}_n\}_{8\times 1}, \quad c_n(z) = -\langle \overline{\overline{D}}_n(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{G}_n\}_{8\times 1}, \\ g_n(z) &= \langle e_n(z)\rangle^{\mathrm{T}}_{1\times 8}\{\overline{G}_n\}_{8\times 1} \end{aligned} \quad (3a-i)$$

$n = 1,..\alpha$

In Eqs. (3a–i), vectors $\{\overline{A}_0\}_{8\times 1}$, $\{\overline{F}_n\}_{8\times 1}$, $\{\overline{G}_n\}_{8\times 1}$ are

$$\begin{aligned} \langle\overline{A}_0\rangle^T_{1\times 8} &= \langle\, \overline{A}_{0,1}\ \ \overline{A}_{0,2}\ \ \overline{A}_{0,3}\ \ \overline{A}_{0,4}\ \ \overline{A}_{0,5}\ \ \overline{A}_{0,6}\ \ \overline{A}_{0,7}\ \ \overline{A}_{0,8}\,\rangle^T \\ \langle\overline{F}_n\rangle^T_{1\times 8} &= \langle\, \overline{F}_{n,1}\ \ \overline{F}_{n,2}\ \ \overline{F}_{n,3}\ \ \overline{F}_{n,4}\ \ \overline{F}_{n,5}\ \ \overline{F}_{n,6}\ \ \overline{F}_{n,7}\ \ \overline{F}_{n,8}\,\rangle^T \\ \langle\overline{G}_n\rangle^T_{1\times 8} &= \langle\, \overline{G}_{n,1}\ \ \overline{G}_{n,2}\ \ \overline{G}_{n,3}\ \ \overline{G}_{n,4}\ \ \overline{G}_{n,5}\ \ \overline{G}_{n,6}\ \ \overline{G}_{n,7}\ \ \overline{G}_{n,8}\,\rangle^T \end{aligned} \quad (4a-c)$$

They consist of integration constants to be determined from the boundary conditions of the problem, and the function vectors $\langle e_0(z)\rangle^{\mathrm{T}}_{1\times 8}$, $\langle \overline{\overline{C}}_0(z)\rangle^{\mathrm{T}}_{1\times 8}$, $\langle \overline{\overline{F}}_0(z)\rangle^{\mathrm{T}}_{1\times 8}$, $\langle \overline{\overline{A}}_n(z)\rangle^{\mathrm{T}}_{1\times 8}$, $\langle \overline{\overline{D}}_n(z)\rangle^{\mathrm{T}}_{1\times 8}$, $\langle e_n(z)\rangle^{\mathrm{T}}_{1\times 8}$ as obtained in [1] are summarized in Appendix A. We recall that the solution fields for the equilibrium equations as developed in [1] were observed to fully uncouple the contribution of mode n from that of mode $m\neq n$, a feature that is exploited in the present formulation.

In the present paper, rather than assuming conventional polynomial functions, we start with the exact solution of the displacement fields (Eqs. (3a–i)) to formulate the exact shape functions. This is done by expressing the vectors of integration constants$\{\overline{A}_0\}_{8\times 1}$, $\{\overline{F}_n\}_{8\times 1}$, $\{\overline{G}_n\}_{8\times 1}$ in terms of the nodal displacements $\langle\Delta_0\rangle^{\mathrm{T}}_{1\times 8}$, $\langle\Delta_{n,1}\rangle^{\mathrm{T}}_{1\times 8}$, $\langle\Delta_{n,2}\rangle^{\mathrm{T}}_{1\times 8}$ defined as

$$\begin{aligned} \langle\Delta_0\rangle^{\mathrm{T}}_{1\times 8} &= \langle a_0(0)\ c_0(0)\ f_0(0)\ f'_0(0)\ a_0(\ell)\ c_0(\ell)\ f_0(\ell)\ f'_0(\ell)\rangle^{\mathrm{T}} \\ \langle\Delta_{n,1}\rangle^{\mathrm{T}}_{1\times 8} &= \langle a_n(0)\ c_n(0)\ f_n(0)\ f'_n(0)\ a_n(\ell)\ c_n(\ell)\ f_n(\ell)\ f'_n(\ell)\rangle^{\mathrm{T}} \\ \langle\Delta_{n,2}\rangle^{\mathrm{T}}_{1\times 8} &= \langle b_n(0)\ d_n(0)\ g_n(0)\ g'_n(0)\ b_n(\ell)\ d_n(\ell)\ g_n(\ell)\ g'_n(\ell)\rangle^{\mathrm{T}} \end{aligned} \quad (5a-c)$$

through

$$\begin{aligned} \{\Delta_0\}_{8\times 1} &= [L_0]_{8\times 8}\{\overline{A}_0\}_{8\times 1} \\ \{\Delta_{n,1}\}_{8\times 1} &= [L_{n,1}]_{8\times 8}\{\overline{F}_n\}_{8\times 1} \\ \{\Delta_{n,2}\}_{8\times 1} &= [L_{n,2}]_{8\times 8}\{\overline{G}_n\}_{8\times 1} \end{aligned} \quad (6a-c)$$

in which matrices $[L_0]_{8\times 8}$, $[L_{n,1}]_{8\times 8}$ and $[L_{n,2}]_{8\times 8}$ are defined in Appendix B. Eqs. (6a–c) are solved for the vectors of integration constants yielding

$$\begin{aligned} \{\overline{A}_0\}_{8\times 1} &= [L_0]^{-1}_{8\times 8}\{\Delta_0\}_{8\times 1} \\ \{\overline{F}_n\}_{8\times 1} &= [L_{n,1}]^{-1}_{8\times 8}\{\Delta_{n,1}\}_{8\times 1} \\ \{\overline{G}_n\}_{8\times 1} &= [L_{n,2}]^{-1}_{8\times 8}\{\Delta_{n,2}\}_{8\times 1} \end{aligned} \quad (7a-c)$$