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Cylindrical membrane partially stretched on a rigid cylinder

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ABSTRACT

We consider the equilibrium problem of a hyperelastic thin-walled tube. One end of the tube is placed over an immovable, rough, rigid cylinder. We assume that the deformation of the tube is finite and axisymmetric. The tube is modeled by a cylindrical membrane. The membrane is composed of an incompressible, homogeneous, isotropic elastic material. We use Bartenev–Khazanovich (Varga) strain energy function. A contact between the membrane and the rigid cylinder is with a dry friction. The membrane will not slide off the cylinder only by a friction and at a sufficient contact area. The friction is described by Coulomb's law. We study a minimum length of the membrane which is in contact with the rigid cylinder and is needed to the equilibrium of the membrane.

1. Introduction

In this study we consider finite axisymmetric deformations of a cylindrical elastic membrane. Finite axisymmetric deformations of thin-walled cylindrical membranes have been studied in many papers. The elastic membrane theory under finite strains was given in [1,2] and others. For axisymmetric deformations of a membrane of revolution the equilibrium equations reduce to ordinary differential equations. To solve obtained non-linear boundary-value problems numerical methods are often used. For homogeneous circular cylindrical membranes, which composed of an isotropic material and subjected to a normal surface load, the Pipkin's first integral is used [3]. In [2] the problem of a superposition of a small deformation on known finite deformation has been considered. In [4,5] the equations governing the incremental state of stress in an orthotropic circular membrane tube have been derived and discussed. These methods can be applied to study a stability of equilibrium states.

The problem of inflation and tension an homogeneous cylindrical membrane with a constant thickness has an explicit solution [2]. In [6–12] non-monotonic "force–deformation" relations and stability of cylindrical membrane composed of hyperelastic materials have been studied under large strains. The stability under overall axial compression of a finitely inflated cylindrical membrane composed of highly elastic material has been investigated in [13]. The effect on stability of the flow of an incompressible fluid through the tube has been considered in [14].

The problem of stretching a cylindrical membrane into an annulus has been studied in [15]. The axially symmetric deformations of a circular cylindrical membrane in which the ends are pulled apart while retaining the radii of the ends fixed have been considered [16,17]. A circular cylindrical membrane subjected to longitudinal extension and twist has been studied in [18,19]. In [19] the effect of radially expanding one end of a cylindrical membrane has been investigated also. In [17] the stationary potential-energy and complementary-energy principles was used to provide upper and lower bounds on a solution of the problem. In [16,18,19] the problems are solved by numerical methods. Wrinkling of the membrane due to twist is taken into account in an approximate way by introducing a relaxed strain energy function in [18,19].

Numerical and experimental analyses of inflation and longitudinal extension have been carried in [20–22]. The inhomogeneous cylindrical membranes have been studied in [9,22-26]. The equilibrium of connecting two sections of different materials or/and different radii has been investigated in [23–25,9]. Local imperfections were considered in [22]. A hyperelastic cylindrical membrane with non-uniform thickness pressurized by internal gas or fluid has been considered in [26].

The finite deformations of an isotropic circular cylindrical membrane subjected to a finite extension and gradually filled with liquid have been investigated both theoretically and experimentally in [27]. The instability and the bifurcation of the equilibrium states of fluid-loaded pre-stretched cylindrical membranes have been studied in [28].

The contact problems of a cylindrical membrane and a solid body are studied few. Finite deformations of a cylindrical

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membrane enclosing a rigid body have been considered in [29–33]. The contact problems of a pressurized cylindrical membrane and a surrounding solid body have been studied in [34–39]. The pressure–distension characteristics of rubber tube enclosed in constraining tubes has been investigated experimentally in [34]. In [36] axisymmetric multiple-contact problems were investigated, together with comparison with experiment. An example of stretch-blow-moulding was also given. We note that most of the authors considered frictionless contact problems. In [39] the cylindrical membrane is inflated and is constrained by a soft substrate. Frictionless and adhesive contacts are modeled during inflation and deflation.

Theory of hyperelastic cylindrical membrane is applied to model nanotubes [40], to create new devices [41,42], in biomechanics [31–33] and others.

In this paper we consider a thin-walled tube composed of hyperelastic material. Its one end is placed over an immovable, rough, rigid cylinder. It can show from the experiment or the equilibrium equations that the tube tends to slide off the rigid cylinder. If the tube does not fixed on the cylinder, then the equilibrium is possible due to friction. It is clear that some contact area is also needed for equilibrium.

We assume that a bending stiffness is neglected. We model the thin-walled tube by a semi-infinite cylindrical membrane. A material of the tube is non-linear elastic, isotropic and incompressible. A Bartenev–Khazanovich model of the material is used. We solve the problem in the framework of the non-linear theory of elastic membranes. We consider an axisymmetric deformation of the cylindrical membrane. We assume that Coulomb's law holds for frictional stresses between the membrane and the rigid cylinder.

The problem reduces to a boundary-value problem for non-linear second-order ordinary differential equations. We derive an explicit solution for an incompressible Bartenev–Khazanovich material. We analyze the effects of a friction coefficient and a radius of the rigid cylinder on the contact area necessary to equilibrate the membrane.

2. Finite axisymmetric deformations of membranes of revolution

Our analysis is based on the so-called direct theory of elastic membranes. To describe configurations of the membrane, we consider a reference surface of revolution, defined parametrically by a vector-valued position function

$$\mathbf{r} = r(s)\mathbf{e}_r + z(s)\mathbf{e}_z,$$

where $\{\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_z\}$ are the unit vectors of the cylindrical coordinate system $\{r, \varphi, z\}$. Let (s, φ) be Lagrangian coordinates which serve as Gaussian coordinates on the reference surface.

We consider deformations that map the reference surface of revolution onto the other surfaces of revolution. The particle (s, φ) at $\mathbf{r}(s, \varphi)$ is displaced to a point with position

$$\mathbf{R} = R(s)\mathbf{e}_r + Z(s)\mathbf{e}_z.$$

The equilibrium equations can be written in the form

$$\frac{dL_1}{ds} + \frac{L_1 - L_2}{R}\frac{dR}{ds} + \xi_1\lambda_1 = 0, \tag{1}$$

$$L_1\kappa_1 + L_2\kappa_2 + \xi = 0, \tag{2}$$

where L_1 and L_2 are the principal stress resultants in the directions of the tangents to the meridian curves and the curves of latitude respectively, κ_1 and κ_2 are the principal curvatures of the deformed surface, λ_1 and λ_2 are the principal stretches, ξ is the surface normal load, ξ_1 is the surface tangential load along a generator.

The principal stretches and curvatures are

$$\lambda_1 = \sqrt{\frac{(R')^2 + (Z')^2}{(r')^2 + (z')^2}}, \quad \lambda_2 = \frac{R}{r},$$

$$\kappa_1 = \frac{\psi'}{\lambda_1}, \quad \kappa_2 = \frac{\sin\psi}{r\lambda_2}, \quad \tan\psi = \frac{Z'}{R'}, \tag{3}$$

where $()' = d()/ds$, ψ is the angle between the tangent to a generator and the vector $\mathbf{e}_r$.

The general strain–energy function for an incompressible material is given by $W = W(\lambda_1, \lambda_2)$. The stress resultants can be written in the form

$$L_1 = \frac{h}{\lambda_2}\frac{\partial W}{\partial \lambda_1}, \quad L_2 = \frac{h}{\lambda_1}\frac{\partial W}{\partial \lambda_2},$$

where h is the uniform initial thickness of the membrane. The thickness H of the deformed membrane is determined by

$$H = \frac{h}{\lambda_1\lambda_2}.$$

Our calculations are based on the Bartenev–Khazanovich (Varga) strain–energy function for isotropic elastic incompressible material [43]. This function has the form

$$W = 2\mu\left(\lambda_1 + \lambda_2 + \frac{1}{\lambda_1\lambda_2} - 3\right), \tag{4}$$

and the stress resultants are given by

$$L_1 = \frac{2\mu h}{\lambda_2}\left(1 - \frac{1}{\lambda_1^2\lambda_2}\right), \quad L_2 = \frac{2\mu h}{\lambda_1}\left(1 - \frac{1}{\lambda_1\lambda_2^2}\right). \tag{5}$$

3. Cylindrical membrane partially stretched on a rigid cylinder

Let the membrane be a semi-infinity right circular cylinder of radius r_0, in its reference configuration (Fig. 1a):

$$r(s) = r_0, \quad z(s) = s, \quad s \in [0, +\infty), \quad \varphi \in [0, 2\pi].$$

We assume that the one end of the membrane partially stretched on a rough, rigid cylinder with radius $R_0 > r_0$ (Fig. 1c). The membrane fits tightly to a side surface of the rigid cylinder. We denote by L the length of the part of the membrane which is in contact with the rigid cylinder, and by $s \in [0, s_L]$ the Lagrangian coordinates of membrane particles which are in contract. We assume that Coulomb's law holds for frictional stresses between the membrane and the rigid cylinder. Surface loads are absent on that part of the surface of the deformed membrane, which is not in contact with the rigid cylinder.

In general, the Coulomb friction is modeled by $|\xi_1| \le f|\xi|$, where f is a coefficient of static friction. The frictional stresses are opposite to the motion that the membrane would experience in the absence of friction. We consider a limit case of equilibrium, such that the frictional stresses in all parts of the contact region reach the limit simultaneously. The membrane tends to slide off the rigid cylinder. So, we assume that the frictional stresses are opposite to the Z-axis and $|\xi_1| = f|\xi|$ in the contact region. We have (Fig. 2a)

$$\xi = \begin{cases} -q(s), & 0 \le s \le s_L, \\ 0, & s > s_L, \end{cases} \quad \xi_1 = \begin{cases} -fq(s), & 0 \le s \le s_L, \\ 0, & s > s_L, \end{cases} \tag{6}$$

We consider the part of the deformed membrane for which $s > s_1 (s_1 \in (s_L, +\infty))$ (Fig. 2b). The force equilibrium equation of this