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A note on characteristic decomposition for two-dimensional Euler system in van der Waals fluids

M. Zafar

Department of Mathematics, Indian Institute of Technology Bombay, Mumbai 400076, India

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ABSTRACT

This paper is devoted to the generalization of a well-known result on reducible equations by Courant and Friedrichs [7] and a motivational result on compressible Euler system within the context of ideal gases by Li et al. [10]. The characteristic decomposition technique has been used to prove that any hyperbolic state, adjacent to a constant state, is simple for a pseudo-steady isentropic irrotational flow, modeled by Euler equations, in van der Waals fluids. Furthermore, this result is extended to full Euler system in self-similar coordinates provided the pseudo-flow characteristics are extending into a constant state.

1. Introduction

Simple waves play a fundamental role in describing and building up solutions of flow problems in gasdynamics [7]. A smooth solution is called simple wave if it depends on a single parameter rather than a pair of parameters. Hyperbolic system in two independent variables,

$$\mathbf{u}_t + A(\mathbf{u})\mathbf{u}_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$ and $n \times n$ matrix $A(\mathbf{u})$ has n real and distinct eigenvalues, can be diagonalized in terms of Riemann invariants for $n=2$. The invariance of Riemann invariants allows us to infer that any flow adjacent to a constant domain is simple; however, system (1) may not be diagonalized in terms of Riemann invariants for $n > 2$ [8]. Moreover the technique of invariance of Riemann invariants, to show that it is a simple flow adjacent to a constant region, breaks down if coefficient matrix A in (1) depends on $(x, t, \mathbf{u})$ rather than $\mathbf{u}$ only. Another important tool, known as characteristic decomposition, to study the hyperbolic system of conservation laws was first discussed by Dai and Zhang [2] for the pressure gradient system and later on by Li et al. [10] in their pioneering work on the compressible Euler system within the context of ideal gases. It is used extensively to overcome the difficulties encountered in the hodograph technique for the interaction of rarefaction waves [3–6]. It helps us to find not only Riemann invariants for homogeneous reducible quasilinear hyperbolic systems but also Riemann variants in some hyperbolic systems [1,4,5]. It is effectively used in providing a passage to derive a priori estimates of solutions [13,14,17]. Furthermore, d'Alembert's formula can be derived by using the idea of characteristic decomposition for one-dimensional wave equation.

The main purpose of this paper is to generalize the well-known theorem of Courant and Friedrichs [7] for reducible systems and a motivational work carried out by Li et al. [10] for ideal gases to the pseudo-steady irrotational Euler system for van der Waals fluids. Here, we have used characteristic decomposition technique, used extensively by Zheng, Jiequan, and their collaborators [1,3–6,11–14,19]. The basic equations of the present study are the Euler equations in two-dimensions which can be written as

$$\begin{aligned}
&(\rho)_t + (\rho u)_x + (\rho v)_y = 0,\\
&(\rho u)_t + (p + \rho u^2)_x + (\rho u v)_y = 0,\\
&(\rho v)_t + (\rho u v)_x + (p + \rho v^2)_y = 0,\\
&(E)_t + (up + uE)_x + (vp + vE)_y = 0,
\end{aligned} \tag{2}$$

where ρ is the fluid density, (u, v) are the fluid velocity components, p is the pressure and $E = \rho e + \rho\left(\frac{u^2+v^2}{2}\right)$ is the total energy density with e being the specific internal energy. In this paper, we consider a polytropic van der Waals fluid having the equation of state of the form [6,9]

$$p(\tau, S) = \frac{K(S)}{(\tau - b)^{\delta+1}} - \frac{a}{\tau^2}, \qquad e = \frac{\left(p + \frac{a}{\tau^2}\right)(\tau - b)}{\delta} - \frac{a}{\tau}, \tag{3}$$

where $\tau = \frac{1}{\rho}$ is the specific volume, S is the specific entropy, $K(S)$ is a positive constant depending on the specific entropy, δ is the

E-mail address: zafar@math.iitb.ac.in

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dimensionless quantity lying in the interval $0 < \delta \le \frac{2}{3}$ ($\delta = \frac{2}{3}$ for monatomic fluid), a and b are fluid dependent parameters representing the attraction between the constituent particles and the compressibility limit of the molecules, respectively. The presence of a and b modifies non-trivially the analysis of Euler equations as the density in the model must be bounded. In the absence of intermolecular force of attraction, i.e., $a = 0$, the equation of state (3) can be seen as a perfect fluid polluted by dusty particles [16]. The thermodynamic variables satisfy the condition $TdS = dh - \tau dp$ with T as temperature and h the specific enthalpy; if τ and S are chosen as independent variables, many calculations of Euler system (2) can be simplified; for instance the speed of sound c is given by $c(\tau, S) = \sqrt{-\tau^2 p'(\tau)} = \sqrt{\left(\frac{K(\delta+1)\tau^2}{(\tau-b)^{\delta+2}} - \frac{2a}{\tau}\right)}$, the parameter b lies in the interval $0 \le b < \tau$, and the energy equation $(2)_4$ may be written as $S_t + uS_x + vS_y = 0$.

2. Steady two-dimensional Euler system for van der Waals fluids

Two-dimensional isentropic irrotational Euler equations for the compressible van der Waals fluids can be expressed as

$$\begin{aligned} &(c^2 - u^2)u_x - 2uvu_y + (c^2 - v^2)v_y = 0, \\ &v_x - u_y = 0, \end{aligned} \tag{4}$$

supplemented by Bernoulli's law

$$\frac{u^2 + v^2}{2} + \frac{K}{\delta}\frac{[(\delta+1)\tau - b]}{(\tau - b)^{\delta+1}} - \frac{2a}{\tau} = C_1, \tag{5}$$

where C_1 is an arbitrary constant. System (4), in vector–matrix notation, can be written as

$$\begin{bmatrix} u \\ v \end{bmatrix}_x + \begin{bmatrix} -\dfrac{2uv}{c^2 - u^2} & \dfrac{c^2 - v^2}{c^2 - u^2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_y = 0, \tag{6}$$

which has the following characteristic equation: $(c^2 - u^2)\lambda^2 + 2uv\lambda + (c^2 - v^2) = 0$. Characteristic form of the system (6) yields $\partial_\pm u + \lambda_\mp \partial_\pm v = 0$ where $\partial_\pm := \partial_x + \lambda_\pm \partial_y$. Since λ_- is a function of u and v, we have

$$\partial_-\lambda_- = \left(\partial_u \lambda_- - \frac{1}{\lambda_+}\partial_v \lambda_-\right)\partial_- u. \tag{7}$$

By direct computation, we have

$$\partial_+\partial_- u = \frac{c^2 - v^2}{c^2 - u^2}\left[\left(\frac{2uv}{c^2 - v^2}\right)_x u_y - \left(\frac{c^2 - u^2}{c^2 - v^2}\right)_x u_x\right] + u_y \partial_+ \lambda_-. \tag{8}$$

In view of

$$(c^2)_x = -\frac{\tau}{\tau - b}\left[\delta + \frac{2b}{\tau} + \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right]\right](uu_x + vu_y),$$

we have

$$\begin{aligned} \left(\frac{2uv}{c^2 - v^2}\right)_x &= \frac{2\tau}{(\tau - b)(c^2 - v^2)^2}\Bigg[vu_x\Bigg(c^2 + \delta u^2 - v^2 + \frac{b}{\tau}(2u^2 + v^2 - c^2) \\ &\quad + \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right]u^2\Bigg) + uu_y\Bigg(c^2 + (\delta + 1)v^2 + \frac{b}{\tau}(v^2 - c^2) \\ &\quad + \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right]v^2\Bigg)\Bigg], \left(\frac{c^2 - u^2}{c^2 - v^2}\right)_x \\ &= -\frac{\tau}{(\tau - b)(c^2 - v^2)^2}\Bigg[uu_x\Bigg(2c^2 + \delta u^2 - (\delta + 2)v^2 - \frac{2b}{\tau}(c^2 - u^2) \\ &\quad + \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right](u^2 - v^2)\Bigg) - vu_y\Bigg(2c^2 - (\delta + 2)u^2 + \delta v^2 \\ &\quad - \frac{2b}{\tau}(c^2 - v^2) - \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right](u^2 - v^2)\Bigg)\Bigg]. \end{aligned} \tag{9}$$

Furthermore, we have

$$\partial_+\lambda_- = \frac{\tau(u\lambda_- - v)^3}{2(\tau - b)c^2\lambda_-\left[\lambda_-(c^2 - u^2) + uv\right]}\left[\delta + 2 + \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right]\right]\partial_+ u. \tag{10}$$

Now using (9), and (10) into (8), we get

$$\begin{aligned} &\frac{1}{\tau}(\tau - b)(c^2 - u^2)(c^2 - v^2)\partial_+\partial_- u \\ &= \Bigg[u\left[2c^2 + \delta u^2 - (\delta + 2)v^2\right]u_x^2 \\ &\quad + \left[-(\delta + 2)v^3 + (3\delta + 2)vu^2 + Z\right]u_x u_y \\ &\quad + \left[2u(c^2 + (\delta + 1)v^2 + \lambda_+ Z)\right]u_y^2 \\ &\quad - \frac{2bu}{\tau}\left[(c^2 - u^2)u_x^2 - 2uvu_x u_y + (c^2 - v^2)u_y^2\right] \\ &\quad + \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right]\Bigg[u(u^2 - v^2)u_x^2 \\ &\quad + \left[2vu^2 + v(u^2 - v^2) + \frac{Z}{\delta + 2}\right]u_x u_y + \left(2uv^2 + \frac{\lambda_+ Z}{\delta + 2}\right)u_y^2\Bigg]\Bigg], \end{aligned} \tag{11}$$

where Z is given by the following relation $Z = \frac{(\delta+2)(c^2-v^2)(uT - cv)^3}{2(c^2-u^2)T(cT-uv)}$; $T = \sqrt{u^2 + v^2 - c^2}$. Factorization of the quadratic form in (11) along λ_- direction yields

$$\begin{aligned} \partial_+\partial_- u = \frac{\tau}{(\tau - b)(c^2 - u^2)(c^2 - v^2)}\Bigg[&u\left[2c^2 + \delta u^2 - (\delta + 2)v^2\right] \\ &(u_x + \alpha u_y) - \frac{2bu}{\tau}(c^2 - u^2) \times (u_x + \beta u_y) \\ &+ \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right]u(u^2 - v^2)(u_x + \mu u_y)\Bigg]\partial_- u, \end{aligned}$$

where $\alpha = \frac{\left[2u(c^2+(\delta+1)v^2+\lambda_+ Z)\right]}{\lambda_- u\left[2c^2+\delta u^2-(\delta+2)v^2\right]}$, $\beta = \frac{(c^2-v^2)}{\lambda_-(c^2-u^2)}$, and $\mu = \frac{\left(2uv^2+\frac{\lambda_+ Z}{\delta+2}\right)}{\lambda_- u(u^2-v^2)}$. In brief, we state the following theorem:

Theorem 1. *There holds the following characteristic decomposition*

$\partial_+\partial_- u = m\partial_- u$, *where*

$$\begin{aligned} m = \frac{\tau}{(\tau - b)(c^2 - u^2)(c^2 - v^2)}\Bigg[&u\left[2c^2 + \delta u^2 - (\delta + 2)v^2\right] \\ &(u_x + \alpha u_y) - \frac{2bu}{\tau}(c^2 - u^2) \times (u_x + \beta u_y) \\ &+ \frac{2a}{(\tau c)^2}\left[(\delta - 1)\tau + 3b\right]u(u^2 - v^2)(u_x + \mu u_y)\Bigg], \end{aligned}$$