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Escape and collision dynamics in the planar equilateral restricted four-body problem

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ABSTRACT

We consider the planar circular equilateral restricted four body-problem where a test particle of infinitesimal mass is moving under the gravitational attraction of three primary bodies which move on circular orbits around their common center of gravity, such that their configuration is always an equilateral triangle. The case where all three primaries have equal masses is numerically investigated. A thorough numerical analysis takes place in the configuration (x, y) as well as in the $(x, \dot{x})$ space in which we classify initial conditions of orbits into four main categories: (i) bounded regular orbits, (ii) trapped chaotic orbits, (iii) escaping orbits and (iv) collision orbits. Interpreting the collision motion as leaking in the phase space we related our results to both chaotic scattering and the theory of leaking Hamiltonian systems. We successfully located the escape and the collision basins and we managed to correlate them with the corresponding escape and collision times of orbits. We hope our contribution to be useful for a further understanding of the escape and collision properties of motion in this interesting dynamical system.

1. Introduction

Over the years several dynamical systems consisting of few bodies have been investigated and various models have been proposed in order to understand and explain the orbital behaviour of realistic celestial systems or as benchmark models where new mathematical theories can be tested. The most extensively studied dynamical system in celestial mechanics is, beyond any doubt, the classical restricted three-body problem, where the third body (test particle) is considered massless so as not to influence the motion of the two primaries which move in Keplerian orbits (circular or elliptical) around their common center of gravity [24,41,42,61,67,68]. The modern applications to space mechanics and dynamics are probably even more cogent than the classical applications. Today numerous aspects in space dynamics are of paramount importance and of great interest. The applications of the restricted three-body problem create the basis of most of the lunar and planetary theories used for launching artificial satellites in the Earth–Moon system and in solar system in general.

In the same vein the restricted four-body problem is quite similar in the sense that the problem deals once more with the motion of an infinitesimal particle under the attraction of three primary bodies [19,20,30,35,51,52,63]. There are many reasons justifying the study of the four-body problem (restricted or not). To begin with, there are many four-body systems in our Solar System which can be approximated, in a first order, by a four-body problem. A characteristic example is the Sun–Jupiter–Saturn system where the fourth body can be a planet, an asteroid or a satellite of Jupiter or Saturn. Another interesting example is the Sun–Earth–Moon system where the fourth body can be a space vehicle [21,27,34,55]. A special case of the Sun–Earth–Moon restricted four-body problem is the bi-circular problem, where the masses of the primary bodies are revolving in a quasi-bi-circular motion [9].

The Sun–Jupiter–Trojan asteroid can also be viewed as a restricted four-body problem, where the primaries are in the particular configuration of an equilateral triangle. This special configuration is known as the planar equilateral restricted four-body problem (PERFBP). Many scientists studied this dynamical system [18,37,38,46,59,60]. [4] investigated the PERFBP with equal masses, while [10] determined the total number of the equilibrium points for any value of the mass parameter and numerically explored their linear stability. They also computed some families of symmetric periodic orbits. Similar results were obtained in [17] but for the case of two equal masses, while the existence of blue sky catastrophe around a specific collinear equilibrium point was presented in [16]. In a recent paper [12] a large number of families of non-symmetric periodic orbits around Jupiter and the Trojan asteroids was found. Moreover, in [5] it was proved that any double collision can be regularized by using a Birkhoff-type transformations.

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The restricted four-body problem has also applications in galactic dynamics. It is known that approximately two-thirds of the 10^{11} stars in our Galaxy belong to multi-stellar systems [48]. In particular, around one-fifth of these stars form triple systems, while a rough estimate suggests that a further one-fifth of these triple systems belongs to quadruple or higher systems, which can be modelled by the four-body problem [36].

A lot of work has been done regarding the equilibrium points of the PERFBP and their stability [7,10,25,33,40,43,44,54,56]. Another interesting issue is the location of periodic orbits in the PERFBP [6,11,17,58]. In recent years many perturbing forces, such as the oblateness, radiation forces of the primaries, Coriolis and centrifugal force, and variation of the masses of the primaries have been included in the study of PERFBP [1,28,29,31,32,43,56].

In this paper we shall try to explore the orbital dynamics in the PERFBP by performing a systematic orbit classification using the numerical methods introduced in the pioneer works of [41,42]. The same numerical methods have also been successfully used in recent similar studies [67–69,47]. The structure of the paper is as follows: In Section 2 we provide a detailed presentation of the principal aspects of the PERFBP. All the computational methods we used in order to determine the character of the orbits are described in Section 3. In the following section, we conduct a thorough numerical investigation revealing the overall orbital structure (bounded regions and basins of escape/collision) of the system and how it is affected by the value of the Jacobi constant. Our paper ends with Section 5, where the discussion and the conclusions of this work are given.

2. Presentation of the mathematical model

Let us describe the basic properties of the PERFBP. We consider three primary bodies with masses m_i, $i = 1, 2, 3$, in a triangular (or Lagrangian) configuration in which the three primaries move in circular orbits in the same plane around their common center of mass. The three bodies are always located at the vertices of an equilateral triangle [64]. The fourth body is known as an infinitesimal mass (or test particle) and it moves in the same plane acting upon the attraction of the three primaries. It is assumed that the mass of the fourth body is so small that its influence on the motion of the primaries is practically negligible.

We adopt a rotating rectangular system whose origin is the center of mass of the primaries which rotates with a uniform angular velocity, so that the centers of the three primaries to be fixed on the (x, y)-plane. Without loss of generality we assign the primary of mass m_1 on the positive x-axis at $C_1 = (x_1, 0)$. Then the other two primaries with masses m_2 and m_3, respectively are located at $C_2 = \left(x_2, \frac{1}{2}\right)$ and $C_3 = \left(x_2, -\frac{1}{2}\right)$, where $x_1 = \sqrt{3}\mu$, $x_2 = -\frac{\sqrt{3}}{2}(1 - 2\mu)$, while μ is the mass parameter. We normalize the units with the supposition that the sum of the masses and the distance between the primaries both be equal to unity. Therefore $m_1 = 1 - 2\mu$ and $m_2 = m_3 = \mu$, so that $m_1 + m_2 + m_3 = 1$.

Regarding the value of the mass parameter there are three limiting cases:

- When $\mu = 0$ we obtain the rotating Kepler's central force problem with $m_1 = 1$ located at the origin of the coordinates.
- When $\mu = \frac{1}{2}$ we obtain the classical circular restricted three-body problem, with two equal masses $m_1 = m_2 = \frac{1}{2}$, which is known as the Copenhagen problem.
- When $\mu = \frac{1}{3}$ we obtain the symmetric case with three primary bodies with masses equal to $\frac{1}{3}$.

In our study we shall consider the last case.

The forces experienced by the test particle in the coordinate system rotating with angular velocity $\omega = 1$ and origin at the center of the mass can be derived from the following total time-independent effective potential function:

$$\Omega(x, y) = \frac{1 - 2\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu}{r_3} + \frac{1}{2}\left(x^2 + y^2\right), \tag{1}$$

where

$$r_1 = \sqrt{(x - x_1)^2 + y^2},$$
$$r_2 = \sqrt{(x - x_2)^2 + \left(y - \frac{1}{2}\right)^2},$$
$$r_3 = \sqrt{(x - x_2)^2 + \left(y + \frac{1}{2}\right)^2}, \tag{2}$$

are the position vectors from the three primaries to the test particle, respectively. Using a synodical system we fixed the position of the primaries in order to eliminate the time-dependence in the potential function.

The scaled equations of motion describing the motion of the test body in the synodical coordinates (x, y) read

$$\Omega_x = \ddot{x} - 2\dot{y} = \frac{\partial \Omega(x, y)}{\partial x},$$
$$\Omega_y = \ddot{y} + 2\dot{x} = \frac{\partial \Omega(x, y)}{\partial y}, \tag{3}$$

where dots denote time derivatives, while the suffixes x and y indicate the partial derivatives of $\Omega(x, y)$ with resect to x and y, respectively. Here it should be noted that Eqs. (3) are invariant under the symmetry

$$\Sigma: \left(t, x, y, \dot{x}, \dot{y}\right) \rightarrow \left(-t, x, -y, -\dot{x}, \dot{y}\right). \tag{4}$$

The dynamical system (3) admits the well known Jacobi integral

$$J(x, y, \dot{x}, \dot{y}) = 2\Omega(x, y) - \left(\dot{x}^2 + \dot{y}^2\right) = C, \tag{5}$$

where $\dot{x}$ and $\dot{y}$ are the velocities, while C is the Jacobi constant which is conserved and defines a three-dimensional invariant manifold in the total four-dimensional phase space. Thus, an orbit with a given value of its energy integral is restricted in its motion to regions in which $C \leq 2\Omega(x, y)$, while all other regions are forbidden to the test body. If the problem is written in canonical coordinates, then the Jacobi integral corresponds to the value of the Hamiltonian and it is known as the total orbital energy. The value of the total orbital energy E is related with the Jacobi constant by $C = -2E$. It should be emphasized that the existence of the Jacobi integral, allows us to study the problem by fixing the energy level of the value of the Jacobi constant.

In the classical restricted three-body problem there are five coplanar equilibrium points [61]. In the PERFBP on the other hand, Ref. [33] proved that the existence as well as the total number of the equilibrium points (collinear and non-collinear) strongly depends on the value of the mass parameter (see also [10]). In our case where all primaries have the same mass $m_1 = m_2 = m_3 = \frac{1}{3}$, the system admits four collinear (on the x-axis) equilibrium points and six non-collinear (off the x-axis) ones. Due to the equality of the masses of the primaries the PERFBP admits a symmetry and the ten equilibrium points lie on the (x, y)-plane symmetrically to the axes of symmetry $y=0$, $y = \sqrt{3}$ and $y = -\sqrt{3}$. Fig. 1 shows the position of the ten equilibrium points along with the centers of the primary bodies, while in Table 1 we provide the exact coordinates of the equilibrium points. All ten equilibrium points are unstable [4,10]. A thorough discussion of the equilibrium points can be found in [8,39,54]. Furthermore, an analytical examination of the stability of the equilibrium points can be found in [15], while a