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The secular equation for non-principal Rayleigh waves in deformed incompressible doubly fiber-reinforced nonlinearly elastic solids

Nguyen Thi Nam [a], Jose Merodio [a,*], Pham Chi Vinh [b]

[a] Department of Continuum Mechanics and Structures, E.T.S. Ing. Caminos, Canales y Puertos, Universidad Politecnica de Madrid, 28040 Madrid, Spain
[b] Faculty of Mathematics, Mechanics and Informatics, Hanoi University of Science, 334, Nguyen Trai Str., Thanh Xuan, Hanoi, Vietnam

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ABSTRACT

The explicit and implicit secular equations for the speed of a (surface) Rayleigh wave propagating in a pre-stressed, doubly fiber-reinforced incompressible nonlinearly elastic half-space are obtained. Hence, the anisotropy is associated with two preferred directions, thereby modelling the effect of two families of fiber reinforcement. One of the principal planes of the primary pure homogeneous strain coincides with the free surface while the surface wave is not restricted to propagate in a principal direction. Results are illustrated with numerical examples. In particular, an isotropic material reinforced with two families of fibers is considered. Each family of fibers is characterized by defining a privileged direction. Furthermore, the fibers of each family are located throughout the half space and run parallel to each other and perpendicular to the depth direction, i.e. the free surface is a plane of symmetry of the anisotropy. The wave speed depends strongly on the anisotropic character of the material model as well as the direction of propagation.

1. Introduction

The purpose of this paper is to extend the analysis of [1] dealing with Rayleigh waves for materials reinforced with one family of fibers to materials reinforced with two families of fibers in the framework of nonlinear elasticity. This is motivated by several factors. First, the use of doubly fiber-reinforced elastic composites is common in engineering applications. In addition, there is a lot of interest in the acoustics of biological soft tissues (see for example, Destrade et al. [2]). Soft biological tissues have been recognized as highly anisotropic due to the presence of collagen fibers [3] and are modeled as orthotropic materials with two families of fibers.

The Rayleigh wave existence and uniqueness problem has been resolved with the aid of the Stroh formalism [4]. Fu and Mielke [5] and Mielke and Fu [6] also have shown the uniqueness of the surface wave speed based on an identity for the surface-impedance matrix. The surface-wave speed can also be obtained from secular equations of implicit as well as explicit form. The explicit secular equations often admit spurious roots that have to be carefully eliminated, as opposed to the numerical methods based on the Stroh formulation or on the surface-impedance matrix. However, the applications of the explicit secular equations are not limited to numerically determine the surface-wave speed. They are also convenient tools to solve the inverse problem that deals with measured values of the wave speed and their agreement with material parameters (see for instance [7,8]). Explicit secular equations have been given by Malischewsky [7] for isotropic solids, Ting [9,10], Ogden and Vinh [11], Vinh and Ogden [12,13], Vinh et al. [1] for anisotropic solids and Vinh [14,15] for pre-stressed media, among others.

We establish a procedure to obtain both the explicit and implicit secular equations of non-principal Rayleigh waves propagating in incompressible, doubly fiber-reinforced, pre-stressed elastic half-spaces. For transversely isotropic materials the explicit secular equation was given in [1] while the implicit one was given in [16]. We build upon these results and use the polarization vector method to get the secular equation in explicit form. The implicit secular equation is obtained from the so-called propagation condition. The latter equation is used to eliminate the spurious roots that arise in the explicit secular equation.

The study of the propagation of Rayleigh-type surface waves in an elastic half-space subject to pre-stress goes back to the pioneering work of Hayes and Rivlin [17] and since then it has attracted the attention of many researchers. There is a lot of interest in using the equations governing infinitesimal motions superimposed on a finite deformation of a nonlinear elastic half-space because it is applicable to several topics. These include: the non-destructive evaluation of prestressed structures before and during loading (see, for example, Makhort [18,19], Hirao et al. [20],

* Corresponding author.
E-mail address: jose.merodio@upm.es (J. Merodio).

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Husson [21], Delsanto and Clark [22], Dyquennoy et al. [23,24], Hu et al. [25]), the acoustics of soft solids with particular attention to the analysis of biological soft tissues (see, for instance, Destrade et al. [2,26,27], Vinh and Merodio [28,29] and references therein), and the (incremental) stability of the free surface of a deformed material (see, for instance, Destrade et al. [30–32]), among others). Indeed, surface waves have been studied extensively in seismology, acoustics, geophysics, telecommunications industry and materials science (see Adams et al. [33]).

In Section 2, the basic constitutive equations associated with this study are presented. This includes the material model as well as the corresponding equations for infinitesimal waves superimposed on a finite deformation consisting of a pure homogeneous strain. In Section 3, the Stroh formalism is applied to the analysis of infinitesimal surface waves propagating in a statically, finitely and homogeneously deformed doubly fiber-reinforced half-space. The free surface is assumed to coincide with one of the principal planes of the primary strain, but a propagating surface wave is not restricted to a principal direction (see [34] for a parallel work that enlightens this analysis). The implicit and explicit secular equations are presented. In Section 4, the results are illustrated numerically in respect of a strain–energy function used to model soft tissue (see [3]).

2. Basic equations

2.1. Kinematics

Consider an elastic body whose reference configuration is denoted by $\mathcal{B}_0$ and a finitely deformed equilibrium configuration. The deformation gradient tensor associated with the deformation is denoted by $\mathbf{F}$. In addition, let (X_1, X_2, X_3) be a fixed rectangular coordinate system in $\mathcal{B}_0$. The precise notation necessary for the analysis will be introduced later on.

Composite materials and some soft tissues are modeled as incompressible isotropic elastic solids reinforced with preferred directions (see [35,36] and references therein). Each preferred direction is associated with a family of parallel fibers. Here, two families of fibers are considered. We denote by $\mathbf{M}$ with components (M_1, M_2, M_3) and $\mathbf{N}$ with components (N_1, N_2, N_3) the unit vectors in these directions in $\mathcal{B}_0$.

The invariants of the right Cauchy–Green deformation tensor, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, where the symbol^T indicates the transpose of a matrix, most commonly used are the *principal* invariants (see, for instance [37]), defined by

$$I_1 = \operatorname{tr}\mathbf{C}, \quad I_2 = \tfrac{1}{2}(I_1^2 - \operatorname{tr}(\mathbf{C}^2)), \quad I_3 = \det\mathbf{C}. \tag{1}$$

The (anisotropic) invariants associated with $\mathbf{M}$ and $\mathbf{C}$ are usually taken as

$$I_4 = \mathbf{M}\cdot(\mathbf{CM}), \quad I_5 = \mathbf{M}\cdot(\mathbf{C}^2\mathbf{M}). \tag{2}$$

For $\mathbf{N}$ and $\mathbf{C}$, the associated invariants are

$$I_6 = \mathbf{N}\cdot(\mathbf{CN}), \quad I_7 = \mathbf{N}\cdot(\mathbf{C}^2\mathbf{N}). \tag{3}$$

Finally, the invariant related to the combination of $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{C}$ is

$$I_8 = \mathbf{M}\cdot(\mathbf{CN})(\mathbf{M}\cdot\mathbf{N}). \tag{4}$$

2.2. Material model

The anisotropic nonlinear elastic strain–energy function W depends on $\mathbf{F}$ through the invariants of the right Cauchy–Green deformation tensor. For incompressible materials, the strain energy function can be written as $W = W(I_1, I_2, I_4, I_5, I_6, I_7, I_8)$ since $I_3 = 1$. If $\mathbf{M}$ and $\mathbf{N}$ are perpendicular then the number of independent invariants is six (see [38] for details). The Cauchy stress is

$$\boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I} = \sum_{i=1, i\neq 3}^{8} W_i \mathbf{F}\frac{\partial I_i}{\partial \mathbf{F}} - p\mathbf{I}, \tag{5}$$

where p is a Lagrange multiplier associated with the incompressibility constraint, the shorthand notations $W_i = \partial W/\partial I_i$, $i = 1, 2, 4, 5, 6, 7, 8$ have been used and $\mathbf{I}$ is the 3×3 identity tensor. The Cauchy stress tensor can be written as

$$\begin{aligned}\boldsymbol{\sigma} = {} & 2W_1\mathbf{B} + 2W_2(I_1\mathbf{I} - \mathbf{B})\mathbf{B} + 2W_4\mathbf{m}\otimes\mathbf{m} \\ & + 2W_5(\mathbf{m}\otimes\mathbf{Bm} + \mathbf{Bm}\otimes\mathbf{m}) + 2W_6\mathbf{n}\otimes\mathbf{n} \\ & + 2W_7(\mathbf{n}\otimes\mathbf{Bn} + \mathbf{Bn}\otimes\mathbf{n}) + W_8(\mathbf{m}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{m})\mathbf{M}\cdot\mathbf{N} - p\mathbf{I},\end{aligned} \tag{6}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, $\mathbf{m}=\mathbf{FM}$, and $\mathbf{n}=\mathbf{FN}$. It follows that, in general, the principal directions of stress and strain do not coincide.

In the biomechanics literature, several strain energy functions given by an isotropic elastic material augmented with the so-called *reinforcing models* can be found. We extend the reinforcing models for one family of fibers (see [36] for complete details) to

$$W = \frac{\mu}{2}(I_1 - 3) + f_1(I_4) + f_2(I_6) + g_1(I_5) + g_2(I_7) + G(I_8) \tag{7}$$

in order to illustrate the results. This strain energy function captures the essential features of the analysis that follows. We want to establish results related to the kinematical properties of the invariants I_4 and I_6 as well as the invariants I_5 and I_7. The results allow us to distinguish the effects of the different invariants. The invariant I_8 is also considered so as to evaluate its influence. For specific details and analysis of the reinforcing models we refer to [35,36]. Here, we just mention that the energy function and the stress must vanish in the reference configuration. In Section 4, we will further make this clear since a certain strain–energy function is used.

2.3. Linearized incremental equations of motion

Consider an incompressible, doubly fiber-reinforced, semi-infinite body $\mathcal{B}$ in its unstrained state $\mathcal{B}_0$ that occupies the region $X_2 \geq 0$. Fibers of each family run parallel to each other and perpendicular to the depth direction X_2, i.e. $M_2 = 0$ and $N_2 = 0$. The body is subjected to a finite pure homogeneous strain with principal directions given by the X_i-axes. A finitely deformed (pre-stressed) equilibrium state $\mathcal{B}_e$ is obtained. A small time-dependent motion is superimposed upon this pre-stressed equilibrium configuration to reach a final material state $\mathcal{B}_t$, called current configuration. The vector components of a representative particle are denoted by X_i, $x_i(\mathbf{X})$, $\tilde{x}_i(\mathbf{X}, t)$ in $\mathcal{B}_0$, $\mathcal{B}_e$ and $\mathcal{B}_t$, respectively. The deformation gradient tensor associated with the deformations $\mathcal{B}_0 \to \mathcal{B}_t$ and $\mathcal{B}_0 \to \mathcal{B}_e$ is denoted by $\bar{\mathbf{F}}$ and $\mathbf{F}$, respectively, and are given in component form by

$$\bar{F}_{iA} = \frac{\partial \tilde{x}_i}{\partial X_A}, \quad F_{iA} = \frac{\partial x_i}{\partial X_A}. \tag{8}$$

It is clear from (8) that

$$\bar{F}_{iA} = (\delta_{ij} + u_{i,j})F_{jA}, \tag{9}$$

where δ_{ij} is the Kronecker operator, $u_i(\mathbf{X}, \mathbf{t})$ denotes the small time-dependent displacement associated with the deformation $\mathcal{B}_e \to \mathcal{B}_t$ and a comma indicates differentiation with respect to the indicated spatial coordinates in $\mathcal{B}_e$.

The necessary equations including the linearized equations of motion for anisotropic incompressible materials are summarized. The incremental components of the nominal stress tensor S_{ji} are