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Peculiarities of wave dynamics in media with oscillating inclusions

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ABSTRACT

The article is concerned with mathematical models for media with oscillating inclusions. These models consist of mutually connected equations, one of which is the wave equation for carrying medium and others are equations of motion for partial oscillators. To close these models, we use cubic and nonlocal equations of state for the carrying medium. Travelling wave solutions to these models are studied in detail. Using qualitative analysis methods, the phase space is shown to contain periodic, homo- and heteroclinic trajectories. Moreover, in the case of nonlocal models we observe the creation of quasi-periodic and chaotic regimes. Bifurcations of localized regimes are studied via the Poincaré section technique.

1. Introduction

In non-equilibrium conditions natural materials begin to manifest hidden properties reflecting peculiarities in their internal structure. Among the most important properties of the medium we distinguish the discreetness of a medium and oscillating dynamics of the structural elements [1,2].

To describe these features in mathematical models for media, the extra volumetric forces causing the movements of the structural elements are incorporated in the continual models [3–5]. We thus consider the structured media as mutually penetrated continua. One of them obeys the wave equation, whereas another one is the oscillating inclusion described by the set of equations for partial oscillators. This leads us to the following equations of motion for structured media:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma}{\partial x} - \sum_{j=1}^{N} m_j \frac{\partial^2 w_j}{\partial t},$$
$$\frac{\partial^2 w_j}{\partial t^2} + \omega_j^2 \left(w_j - u \right) = 0, \quad j \in 1, \ldots, N, \tag{1}$$

where ρ is the density of the carrying medium, σ is the stress; $u(x, t)$, $w_j(x, t)$ are the displacements of the bulk medium and a typical oscillator with the natural frequency ω_j; $m_j\rho$ is the density of oscillating inclusions. In this report we restrict our consideration to the case when $N=1$, i.e. only one type of oscillators is taken into account.

But, under high-intense impulse loading, the structured substances often manifest strong nonlinear effects. Moreover, when the medium is far from equilibrium, various relaxing processes within the elements of the structure take place and then the linear model should be supplemented by physical nonlinearity and nonlocal effects. This can be displayed in the corresponding equations of state for the carrying media closing model (1). Within the framework of the presented models we are going to study travelling wave regimes and their bifurcations when the parameters of nonlinearity and nonlocality are varied.

In Section 2 we classify the wave solutions of model (1) with the cubic equation of state. These solutions are described by the planar Hamiltonian dynamical system which admits detailed explorations via qualitative analysis methods. Section 3 deals with the model incorporating the spatio-temporal nonlocal effects. Using the Poincaré section technique, the localized attractors in the four dimensional dynamical system were investigated.

2. Wave solutions to model (1) with the cubic equation of state

We are interested in the travelling wave solutions having the form

$$u = U(s), \quad w = W(s), \quad s = x - Dt, \tag{2}$$

where D is the constant wave velocity.

Let us consider model (1) with the cubic equation of state

$$\sigma = e_1 \frac{\partial u}{\partial x} + e_2 \left(\frac{\partial u}{\partial x} \right)^2 + e_3 \left(\frac{\partial u}{\partial x} \right)^3. \tag{3}$$

After the substitution of expression (2) into (1) and (3), system (1) reads

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$$D^2U' = \sigma - D^2mW', \tag{4}$$

$$\sigma = e_1U' + e_2U'^2 + e_3U'^3, \tag{5}$$

$$W'' + \Omega^2\left(W - U\right) = 0, \tag{6}$$

where $\Omega = \omega D^{-1}$.

From (4) and (5) it follows

$$W' = \alpha_1 R + \alpha_2 R^2 + \alpha_3 R^3, \quad R = U', \tag{7}$$

where $\alpha_1 = \frac{e_1 - D^2}{mD^2}$, $\alpha_2 = \frac{e_2}{mD^2}$, $\alpha_3 = \frac{e_3}{mD^2}$.

Excluding W' from (6) with the help of (7) leads to the following dynamical system:

$$R' = Z,$$
$$Z' = -\frac{\left(6\alpha_3 R + 2\alpha_2\right)Z^2 + \Omega^2\left(\left(\alpha_1 - 1\right)R + \alpha_2 R^2 + \alpha_3 R^3\right)}{\alpha_1 + 2\alpha_2 R + 3\alpha_3 R^2}. \tag{8}$$

System (8) has the fixed points

$$O(0;\,0), \quad A_\pm = \left(\frac{-\alpha_2 \pm \sqrt{\alpha_2^2 - 4\alpha_3\left(\alpha_1 - 1\right)}}{2\alpha_3};\,0\right).$$

Analysing the stability of fixed points in the linear approximation, it is easy to verify that the fixed point O is characterized by the eigenvalues $\lambda^2 = \Omega^2\frac{1-\alpha_1}{\alpha_1}$, $\alpha_1 \neq 0$. Therefore, at $0 < \alpha_1 < 1$ the origin is a saddle, whereas in other cases it is a centre.

Case 1: At first, consider the case $\alpha_1 = 0$. Then, instead of system (8), it is preferable to deal with the following dynamical system:

$$R' = Z,$$

$$R\left(2\alpha_2 + 3\alpha_3 R\right)Z' = -\left(Z^2\left(6\alpha_3 R + 2\alpha_2\right) + \Omega^2\left(-R + \alpha_2 R^2 + \alpha_3 R^3\right)\right), \tag{9}$$

which has the same fixed points as system (8). In the vicinity of fixed points $A_\pm$ the linearized matrices $J^\pm$ have the form

$$J^\pm = \begin{pmatrix} 0 & 1 \\ J_{21}^\pm & 0 \end{pmatrix},$$

where

$$J_{21}^\pm = -\frac{2\Omega^2\alpha_2^2\left(3 + 12k \pm \mathrm{sgn}\left(\alpha_2\right)\sqrt{1+4k}\right)}{\left(\alpha_2 \pm 3\sqrt{\alpha_2^2 + 4\alpha_3}\right)^2},$$
$$k = \frac{\alpha_3}{\alpha_2^2}. \tag{10}$$

It is clear that the eigenvalues of matrices $J^\pm$ satisfy the equation $\lambda^2 - J_{21}^\pm = 0$ and, consequently, the type of points $A_\pm$ stability depends on the sign of the following expression:

$$\Delta^\pm = 3 + 12k \pm \mathrm{sgn}\left(\alpha_2\right)\sqrt{1+4k}$$
$$= \sqrt{1+4k}\left(3\sqrt{1+4k} \pm \mathrm{sgn}\left(\alpha_2\right)\right).$$

So, from the analysis of Δ^+ it follows that if $\alpha_2 > 0$ and $k > -1/4$ point A_+ is a centre. If $\alpha_2 < 0$ and $-1/4 < k < -2/9$, then point A_+ is a saddle, and, finally, for $k > -2/9$ point A_+ is a centre. Considering point A_-, one should change the signs of inequalities for α_2 to opposite ones.

In fact, we can classify the phase planes of the dynamical system (9) varying parameter k only. For instance, assume $\alpha_2 = -1 < 0$ and consider the phase planes when $k \in (-1/4;\,-2/9) \cup \{-2/9\} \cup (-2/9;+\infty)$.

For $k > -2/9$, both points $A_\pm$ are centres and the asymptotes $R=0$ and $R = -2\alpha_2/3\alpha_3$ are observed. It is easy to be convinced that for $k > 0$ the fixed points $A_\pm$ lie in the different half planes (Fig. 1a), whereas $A_\pm$ lie in the negative half plane if $-2/9 < k < 0$ (Fig. 1b). Moreover, the asymptote $R = -2\alpha_2/3\alpha_3$ lies between the fixed points for proper k. Due to the existence of the asymptotes and the closed trajectories passing through the fixed points, non-analytic solutions can be observed. In particular, the periodic trajectories passing through the origin (curve 4 Fig. 1b) correspond to compactons, i.e. wave solutions with a compact support [6,7]. Other non-analytic solutions can be constructed by means of combining the curves 3–2 or 6–5–7–2 (Fig. 1b). The schematic profiles of $R(s)$ for these non-analytic solutions are depicted in Fig. 2.

If we choose the parameter k from the interval $(-1/4;\,-2/9)$, then the phase plane (Fig. 1c) contains the saddle A_- whose separatrices form a homoclinic loop, and the centre A_+. As in the previous case, we can construct several non-smooth regimes like 1–2–3, 2–6–7–8, 4–5–2.

At $k = -2/9$ we see (Fig. 1d) an exotic situation where the phase trajectory goes through two fixed points separating smooth and non-smooth periodic regimes.

Case 2: Now assume that $\alpha_1 \neq 0$. Then the trajectory $R=0$ is not singular. System (8) has three fixed points O, $A_\pm$ and two asymptotes $R_\pm = \frac{-\alpha_2 \pm \sqrt{\alpha_2^2 - 3\alpha_3\alpha_1}}{3\alpha_3}$. Typical phase planes, in general, have been presented in [8].

To supplement phase portraits analysis, we use the Hamiltonian for system (8)

$$H = \frac{Z^2}{2}\left(\alpha_1 + 2\alpha_2 R + 3\alpha_3 R^2\right)^2 + \frac{(\Omega R)^2}{12}\Bigg(6\left(\alpha_1 - 1\right)\alpha_1 - 4\left(2 - 3\alpha_1\right)\alpha_2 R$$
$$+ 3\left(2\alpha_2^2 + \left\{4\alpha_1 - 3\right\}\alpha_3\right)R^2 + 12\alpha_2\alpha_3 R^3 + 6\alpha_3^2 R^4\Bigg).$$

Using this function, the analytical expression describing the homoclinic loop of the saddle O has the following form:

$$Z^2 = -\frac{(\Omega R)^2}{6\left(\alpha_1 + 2\alpha_2 R + 3\alpha_3 R^2\right)^2}\Bigg(6\left(\alpha_1 - 1\right)\alpha_1 - 4\left(2 - 3\alpha_1\right)\alpha_2 R$$
$$+ 3\left(2\alpha_2^2 + \left\{4\alpha_1 - 3\right\}\alpha_3\right)R^2 + 12\alpha_2\alpha_3 R^3 + 6\alpha_3^2 R^4\Bigg). \tag{11}$$

Note that the function $Z(R)$ is not an even one due to the presence of terms with α_2. The asymmetry of phase space associated with α_2 is encountered in studies of materials exhibiting the different tensile and compressive responses [9].

Let us now estimate the maximal deviation $R_{\max}$ from the fixed point O and consider its dependence on wave velocity D. Analysing the curves defined by (11), one can conclude that the function $Z(R)$ undergoes maximal deviation from O in the points where the homoclinic trajectory intersects the axis $Z=0$. Therefore, $R_{\max}$ obeys the equation

$$6\left(\alpha_1 - 1\right)\alpha_1 - 4\left(2 - 3\alpha_1\right)\alpha_2 R_{\max} + 3\left(2\alpha_2^2 + \left\{4\alpha_1 - 3\right\}\alpha_3\right)R_{\max}^2$$
$$+ 12\alpha_2\alpha_3 R_{\max}^3 + 6\alpha_3^2 R_{\max}^4 = 0.$$

Assuming that $\alpha_2 = \varepsilon \ll 1$, this equation can be reduced to the equation

$$R_{\max}^4 + \mu_3\varepsilon R_{\max}^3 + \left(\mu_2 + \varepsilon a\right)R_{\max}^2 + \mu_1\varepsilon R_{\max} + \mu_0 = 0, \tag{12}$$

where $\mu_3 = \frac{2}{\alpha_3}$, $\mu_2 = \frac{4\alpha_1 - 3}{2\alpha_3}$, $a = \frac{1}{\alpha_3^2}$, $\mu_1 = \frac{2(3\alpha_1 - 2)}{3\alpha_3^2}$, $\mu_0 = \frac{(\alpha_1 - 1)\alpha_1}{\alpha_3^2}$. We derive the approximate solution of (12) in the form