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Dynamical analysis of Mathieu equation with two kinds of van der Pol fractional-order terms

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ABSTRACT

In this paper the dynamics of Mathieu equation with two kinds of van der Pol (VDP) fractional-order terms is investigated. The approximately analytical solution is obtained by the averaging method. The steady-state solution, existence conditions and stability condition for the steady-state solution are presented, and it is found that the two kinds of VDP fractional coefficients and fractional orders remarkably affect the steady-state solution, which is characterized by the additional damping coefficient (ADC) and additional stiffness coefficient (ASC). The comparisons between the analytical and numerical solutions verify the correctness and satisfactory precision of the approximately analytical solution. The presented typical amplitude–frequency curves illustrate the important effects of two kinds of VDP fractional-order terms on system dynamics. The application of two VDP fractional-order terms in vibration control is discussed. At last, the detailed results are summarized and the conclusions are made.

1. Introduction

Although fractional calculus had been proposed for more than 300 years, its applications to physics and engineering were just a recent research focus [1,2]. Comparing with the traditional integer-order counterpart, the fractional-order system is much closer to the real nature of the world, and has more advantages, such as strong ability of anti-noise, good robustness, high control precision and so on. The fractional derivatives are an adequate tool to model the frequency-dependent damping behavior of materials and physical systems, and which have played a very important role in various fields such as viscoelasticity, electrochemistry, bioengineering, mechanics, automatic control and signal processing. Accordingly, a lot of researchers in some relevant fields had applied the fractional-order models to solve the problems they met. For example, Gorenflo et al. [3], Jumarie et al. [4], Ishteva [5] and Agnieszka [6] et al. respectively studied the definitions and numerical methods of fractional-order calculus under different senses. Wang and Hu et al. [7,8] investigated a linear single degree-of-freedom oscillator with damping force of fractional-order derivative, and obtained the composition of the solution without external excitation. Shen et al. [9–12] investigated several linear and nonlinear fractional-order oscillators by averaging method, and found that the fractional-order derivatives had both damping and stiffness effects on the dynamical response in those oscillators. Li et al. [13] discussed the properties of three kinds of fractional derivatives, and the sequential property of the Caputo derivative was also derived. Li et al. [14,15] had done a lot of researches in the mathematical theory of fractional-order calculus, and also established some efficient numerical algorithms. Wahi and Chatterjee [16] studied a special linear single degree-of-freedom oscillator with fractional-order derivative by average method, and analyzed the effects of the fractional-orders derivative. Xu and Li [17] combined Lindstedt–Poincare method with multi-scale method to study fractional-order Duffing oscillator subject to random excitation. Chen and Zhu [18–20] studied some nonlinear fractional-order system with different kinds of noise, and obtained some important statistic properties of the fractional-order system. Caputo et al. [21] presented a new definition of fractional-order derivative which took on two different representations for the temporal and spatial variables. Atangana [22–26] studied new fractional-order derivatives in typical nonlinear equations, and derived some new results about fractional-order derivatives.

The well-known Mathieu equation is a linear differential equation with periodic coefficients, and had been applied in physics and engineering fields. Many scholars had studied the Mathieu equation and found that the fractional-order system could generate different dynamical properties from the integer-order counterpart [27–29]. Van der pol (VDP) oscillator could model the typical self-excited or self-sustained oscillation. Leung et al. [30], Sardar et al. [31], Xie and Lin [32] studied different type of

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fractional-order VDP oscillators by different methods, and found some other important dynamical behaviors. Mathieu-VDP equation occurs in many physics and engineering fields, and it has complex dynamical properties. In recent years, many scholars had studied the Mathieu-VDP equation. For example, Momeni et al. [33] proposed a Mathieu-VDP nonlinear equation to govern the dust grain dynamics in the vicinity of resonance, and finally numerically solved by a fourth-order Runge–Kutta method. Pandey et al. [34] investigated the dynamics of a Mathieu-VDP equation and a Mathieu-VDP-Duffing equation, which was forced both parametrically and nonparametrically. It had been shown that the steady-state response could consist of either 1:1 frequency locking, or quasiperiodic motion. Kalas et al. [35] studied the generalized Mathieu-VDP equation with a small parameter, and the existence of periodic and quasiperiodic solutions was proved by the averaging method and phase space analysis of a derived autonomous equation. Belhaq and Fahsi [36] studied the frequency-locking area of 2:1 and 1:1 resonances in a fast harmonically excited VDP-Mathieu-Duffing oscillator. An averaging technique over the fast excitation was used to derive an equation governing the slow dynamic of the oscillator. Veerman and Verhulst [37] analyzed the VDP-Mathieu equation near and at 1:2 resonance by averaging method, and proved the existence of stable and unstable periodic solutions near the parametric resonance frequency.

In this study, we shall consider the Mathieu equation with two kinds of fractional-order VDP terms by averaging method, where the fractional-order derivatives are classified based on their range and could cover all the cases. The paper is organized as follow. Section 2 presents the approximately analytical solution of the Mathieu equation with two kinds of fractional-order VDP terms, where the effects of the two kinds of fractional-order VDP terms are formulated as additional damping coefficient (ADC) and additional stiffness coefficient (ASC). In Section 3 the steady-state solution and the stability condition of the steady-state solution are analyzed. In Section 4, the comparisons between the approximately analytical solution and the numerical one verify the correctness and satisfactory precision of the analytical solution. Moreover, the effects of the parameters in the two kinds of fractional-order VDP terms are also given in this section. At last, the detail results are summarized and the conclusions are made.

2. Approximately analytical solution of Mathieu equation with two kinds of VDP fractional-order terms

In this paper, we shall consider the Mathieu equation with two kinds of van der Pol (VDP) fractional-order terms as

$$\ddot{x} + 2\zeta\dot{x} + (\delta + 2B\cos\gamma t)x + K_1(x^2-1)D^{p_1}[x(t)] + K_2(x^2-1)D^{p_2}[x(t)] = 0, \tag{1}$$

where 2ζ, δ, and $2B\cos\gamma t$ are the system linear damping coefficient, constant stiffness coefficient and the periodic time-varying stiffness coefficient respectively. $K_1(x^2-1)D^{p_1}[x(t)]$ and $K_2(x^2-1)D^{p_2}[x(t)]$ are two kinds of VDP fractional-order terms, where the fractional orders are restricted as $0 < p_1 < 1$ and $1 < p_2 < 2$, and $K_1(K_1 > 0)$ and $K_2(K_2 > 0)$ are the fractional coefficients of two kinds of fractional-order terms. The introduction of fractional-order terms in Mathieu equation lies in some dynamical devices could be modeled by fractional-order derivative, such as viscoelastic and fluid–solid-coupling device. There are several definitions for fractional-order derivative, such as Grünwald–Letnikov, Riemann–Liouville and Caputo definitions [1,2]. Under wide senses, they are equivalent for most mathematical functions. Accordingly, Caputo's definition is adopted with the form as

$$D^p[x(t)] = \frac{1}{\Gamma(n-p)}\int_0^t \frac{x^{(n)}(u)}{(t-u)^{p-n+1}}\mathrm{d}u, \tag{2}$$

where $\Gamma(y)$ is Gamma function satisfying $\Gamma(y+1) = y\Gamma(y)$, and the fractional order meets $n-1 < p < n$ while n is a natural number.

Using the following transformations of coordinates

$$\zeta = \varepsilon\mu,\ \delta = \omega_0^2,\ B = \varepsilon\beta,\ \gamma = 2\omega,\ K_1 = \varepsilon k_1,\ K_2 = \varepsilon k_2$$

Eq. (1) becomes

$$\ddot{x} + 2\varepsilon\mu\dot{x} + (\omega_0^2 + 2\varepsilon\beta\cos 2\omega t)x + \varepsilon k_1(x^2-1)D^{p_1}[x(t)] + \varepsilon k_2(x^2-1)D^{p_2}[x(t)] = 0. \tag{3}$$

We shall consider the parametric excitation frequency to be $\gamma = 2\omega_0 + \varepsilon\sigma$, where $\varepsilon << 1$ is a small real parameter, i.e. $\omega = \omega_0 + \varepsilon\sigma/2$. In order to obtain the first approximate solution, it could also be rewritten as $\omega^2 = \omega_0^2 + \varepsilon\sigma\omega_0$. So Eq. (3) becomes

$$\ddot{x} + \omega^2 x = \varepsilon\{\sigma\omega_0 x - 2\mu\dot{x} - (2\beta\cos 2\omega t)x - k_1(x^2-1)D^{p_1}[x(t)] - k_2(x^2-1)D^{p_2}[x(t)]\} \tag{4}$$

The solution for Eq. (4) could be supposed as

$$x = a\cos\varphi, \tag{5a}$$

$$\dot{x} = -a\omega\sin\varphi, \tag{5b}$$

where $\varphi = \omega t + \theta$. Based on the averaging method [38,39], one could obtain the standard equations as

$$\dot{a} = -\frac{1}{T\omega}\int_0^T [P_1(a,\theta) + P_2(a,\theta) + P_3(a,\theta)]\sin\varphi\,\mathrm{d}\varphi, \tag{6a}$$

$$a\dot{\theta} = -\frac{1}{T\omega}\int_0^T [P_1(a,\theta) + P_2(a,\theta) + P_3(a,\theta)]\cos\varphi\,\mathrm{d}\varphi, \tag{6b}$$

where

$$P_1(a,\theta) = \varepsilon[\sigma\omega_0 a\cos\varphi + 2\mu a\omega\sin\varphi - (2\beta\cos 2\omega t)a\cos\varphi],$$
$$P_2(a,\theta) = -\varepsilon k_1[(a\cos\varphi)^2 - 1]D^{p_1}[a\cos\varphi],$$
$$P_3(a,\theta) = -\varepsilon k_2[(a\cos\varphi)^2 - 1]D^{p_2}[a\cos\varphi].$$

From the averaging method, one could select the time terminal T as $T = 2\pi$ when $P_i(a,\theta)(i = 1, 2, 3)$ is periodic function, or $T = \infty$ when $P_i(a,\theta)(i = 1, 2, 3)$ is aperiodic one. Expanding the integral in Eq. (6), one could obtain

$$\dot{a} = \dot{a}_1 + \dot{a}_2 + \dot{a}_3, \tag{7a}$$

$$a\dot{\theta} = a\dot{\theta}_1 + a\dot{\theta}_2 + a\dot{\theta}_3, \tag{7b}$$

The first part in Eq. (7) could be simplified as

$$\dot{a}_1 = -\frac{1}{2\pi\omega}\int_0^{2\pi} P_1(a,\theta)\sin\varphi\,\mathrm{d}\varphi = \frac{\varepsilon a\beta\sin 2\theta}{2\omega} - \varepsilon\mu a, \tag{8a}$$

$$a\dot{\theta}_1 = -\frac{1}{2\pi\omega}\int_0^{2\pi} P_1(a,\theta)\cos\varphi\,\mathrm{d}\varphi = \frac{\varepsilon a\beta\cos 2\theta}{2\omega} - \frac{\varepsilon a\sigma\omega_0}{2\omega}. \tag{8b}$$

In order to calculate the second part and the third part for Eq. (7)

$$\dot{a}_2 = -\lim_{T\to\infty}\frac{1}{T\omega}\int_0^T P_2(a,\theta)\sin\varphi\,\mathrm{d}\varphi = \lim_{T\to\infty}\frac{\varepsilon k_1}{T\omega}\int_0^T [(a\cos\varphi)^2 - 1]D^{p_1}[a\cos\varphi]\sin\varphi\,\mathrm{d}\varphi, \tag{9a}$$