International Journal of Non-Linear Mechanics 74 (2015) 1–6

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

# Local and nonlocal conserved vectors of the system of two-dimensional generalized inviscid Burgers equations

Muhammad Alim Abdulwahhab

*Deanship of Educational Services, Qassim University, Saudi Arabia*

ARTICLE INFO

*Article history:*
Received 3 September 2014
Received in revised form
17 March 2015
Accepted 18 March 2015
Available online 30 March 2015

*Keywords:*
Conserved vectors
First integrals
Non-linear self-adjointness
Noether's theorem
2D inviscid Burgers equation

ABSTRACT

The concept of non-linear self-adjointness for the construction of conservation laws has attracted a lot of interest in recent years. The most noteworthy aspect of it is the likelihood of explicitly constructing the conservation laws for any arbitrary systems of differential equations, in particular for those for which Noether's theorem is not applicable. In this study, we shall use both Noether's theorem and the non-linear self-adjoint method to construct local and nonlocal conserved vectors of the system of two-dimensional Burgers equations under consideration. The first integrals obtained not only give more credence to obtained results due to their generality with respect to any arbitrary functions of the velocity components but are also independent, nontrivial and infinitely many.



## 1. Introduction

We consider the system of two-dimensional generalized inviscid Burgers equations

$$F_1 \equiv u_t + g(u)u_x + f(v)u_y = 0,$$
$$F_2 \equiv v_t + g(u)v_x + f(v)v_y = 0, \tag{1}$$

where $g(u)$ and $f(v)$ are assumed to be respective smooth functions of the velocity components $u = u(x, y, t)$ and $v = v(x, y, t)$ with non vanishing first derivatives. When pressure ($P$) and viscosity ($\mu$) are completely negligible, Eq. (1) becomes the generalized inviscid form of the two-dimensional system of Navier–Stokes-type equations

$$u_t + uu_x + vu_y = -P_x + \mu\nabla^2 u + 2\mu_x u_x + \mu_y(u_y + v_x),$$
$$v_t + uv_x + vv_y = -P_y + \mu\nabla^2 v + 2\mu_y v_y + \mu_x(u_y + v_x), \tag{2}$$

whose group classification was carried out in [1]. Complete group analysis of the Burgers system (1) was first carried out in [2]. In that paper, I used the Lie group method to derive optimal system of one-dimensional subalgebras and subsequently used them to obtain generalized distinct group-invariant solutions of the Burgers system (1). These generalized solutions can be used to study the interactions of the (spatial) velocity components, at different times $t$, in a superfluid flow at zero-temperature limit. They can also be used as bases for the testing of algorithms and computer programs of the numerical solutions of the Burgers system (1) in particular, and all its equivalent equations of mathematical physics in general.

The application of Lie transformations group theory for the construction of invariant solutions of partial differential equations (PDEs) is one of the most active fields of research in the theory of PDEs and applications. The importance of obtaining symmetry generators of any PDE needs no mentioning, it is well known and fully documented in the literatures. Their connection with the construction of conserved vectors of PDEs, proved by Emmy Noether in her celebrated 1918 paper, is what uncovered the fundamental justification for conservation laws. Noethers theorem [3], one of the most amazing and useful theorems in physics, shows that for each variational symmetry generator there is a corresponding conservation law. This allows physicists to gain powerful insights into any general theory in physics just by analyzing the various transformations that would make the form of the laws involved invariant. For instance, the invariance of physical systems with respect to spatial translation, rotation, and time translation respectively give rise to the well-known conservation laws of linear momentum, angular momentum and energy. Various generalizations of Noether's method have been proposed in the literature [4–10] so as to overcome its limitations. These generalizations enable first integrals of almost any PDE (or their system) to be constructed, and thus open ways to understanding their various physical properties.

The purpose of this paper is to construct nontrivial local and nonlocal first integrals of Eq. (1) using both Noether's theorem [3] and the method of non-linear self-adjointness [7]. This is to enable the computation of a wide range of constants of motion of Eq. (1) achievable as can be seen from [11] where the authors used the methods in [6,10] to construct invariant quantities of foam-drainage equation.

*E-mail addresses:* m.wahab@qu.edu.sa, mwahabs@outlook.com

http://dx.doi.org/10.1016/j.ijnonlinmec.2015.03.006
0020-7462/

## 2. Adjoint system and non-linear self-adjointness

The concept of non-linear self-adjointness was introduced by Ibragimov in [7] to extend the applicability of the new conservation laws theorem that he earlier established in [6]. Since its inception about three years ago, many researchers have used it to construct conservation laws for various PDEs of interest in sciences and engineering. Although the concept is analogous to that of Multiplier Approach [12] (see also [13]), the difference is in how the conserved vectors are found afterwards. The non-linear self-adjointness method associate a conservation law with any group of Lie, Lie-Bäcklund or nonlocal symmetries. Thus, one is always guaranteed of obtaining nontrivial conserved vector(s) through one of the symmetry generators of the underlying equation (system). Even if an underlying equation (system) is not non-linearly self-adjoint, one is still assured of obtaining nonlocal conserved vectors. In what follows, we establish the non-linear self-adjointness of the Burgers system (1) which can be used for obtaining local conservations laws.

In accordance with [6], the formal Lagrangian for the Burgers system (1) is given by

$$\mathcal{L} = \alpha\left(u_t + g(u)u_x + f(v)u_y\right) + \beta\left(v_t + g(u)v_x + f(v)v_y\right) \tag{3}$$

where $\alpha$ and $\beta$ are the new dependent variables.

The adjoint system for the Burgers system (1) is defined as

$$F_1^* = \frac{\delta(\mathcal{L})}{\delta u} \equiv \frac{\partial(\mathcal{L})}{\partial u} - D_t\frac{\partial(\mathcal{L})}{\partial u_t} - D_x\frac{\partial(\mathcal{L})}{\partial u_x} - D_y\frac{\partial(\mathcal{L})}{\partial u_y} = 0,$$
$$F_2^* = \frac{\delta(\mathcal{L})}{\delta v} \equiv \frac{\partial(\mathcal{L})}{\partial v} - D_t\frac{\partial(\mathcal{L})}{\partial v_t} - D_x\frac{\partial(\mathcal{L})}{\partial v_x} - D_y\frac{\partial(\mathcal{L})}{\partial v_y} = 0, \tag{4}$$

where $D_t$, $D_x$ and $D_y$ are the total derivatives with respect to $t$, $x$ and $y$ respectively.

Taking into account Eqs. (3) and (4), we obtain the following adjoint system:

$$\alpha_t + g\alpha_x + f\alpha_y + \alpha f' v_y - \beta g' v_x = 0,$$
$$\beta_t + g\beta_x + f\beta_y + \beta g' u_x - \alpha f' u_y = 0. \tag{5}$$

**Definition 2.1.** The two-dimensional Burgers system (1) is said to be (strictly) self-adjoint if the system of equations obtained from the adjoint system (5) after the substitutions $\alpha = u, \beta = v$ (or vice versa) is identical with the original system (1). That is, if

$$F^*\big|_{\substack{\alpha = u \\ \beta = v}} = \zeta(x, u, \ldots)F \tag{6}$$

for some indeterminate variable coefficient $\zeta$.

**Definition 2.2.** The two-dimensional Burgers system (1) is said to be quasi self-adjoint if the system of equations obtained from the adjoint system (5) after the substitutions $\alpha = \varsigma(u,v), \beta = \sigma(u,v)$ is identical with the original system (24) where $\varsigma'(u,v) \neq 0, \sigma'(u,v) \neq 0$ are arbitrary functions. That is, if

$$F^*\big|_{\substack{\alpha = \varsigma(u,v) \\ \beta = \sigma(u,v)}} = \zeta(x, u, \ldots)F \tag{7}$$

for some indeterminate variable coefficient $\zeta$.

The conditions on the arbitrary functions in Definition 2.2 can be replaced with $\varsigma(u,v) \neq 0, \sigma(u,v) \neq 0$.

**Definition 2.3.** The two-dimensional Burgers system (1) is said to be non-linearly self-adjoint if the system of equations obtained from the adjoint system (5) after the substitutions $\alpha = \varsigma(x,y,t,u,v), \beta = \sigma(x,y,t,u,v)$ is identical with the original system (1) where $\varsigma(x,y,t,u,v) \neq 0, \sigma(x,y,t,u,v) \neq 0$ are arbitrary functions. That is, if

$$F^*\big|_{\substack{\alpha = \varsigma(x,y,t,u,v) \\ \beta = \sigma(x,y,t,u,v)}} = \zeta(x, u, \ldots)F \tag{8}$$

for some indeterminate variable coefficient $\zeta$.

Calculation shows that Eq. (1) did not satisfy[1] Definitions 2.1 and 2.2 meaning that it is neither strictly self-adjoint nor quasi self-adjoint. To see whether Definition 2.3 is satisfied, we let

$$\alpha = \varsigma(x,y,t,u,v), \quad \beta = \sigma(x,y,t,u,v) \tag{9}$$

be any arbitrary smooth point-wise substitutions having derivatives

$$\alpha_k = \varsigma_k + \varsigma_u u_k + \varsigma_v v_k,$$
$$\beta_k = \sigma_k + \sigma_u u_k + \sigma_v v_k, \tag{10}$$

for $k = x, y, t$. Using (9) and (10) in Eq. (5), the non-linearly self-adjoint condition (8) become

$$\varsigma_t + \varsigma_u u_t + \varsigma_v v_t + g(u)(\varsigma_x + \varsigma_u u_x + \varsigma_v v_x) + f(v)(\varsigma_y + \varsigma_u u_y + \varsigma_v v_y) + \varsigma f'(v)v_y - \sigma g'(u)v_x$$
$$= \zeta^1(u_t + g(u)u_x + f(v)u_y) + \zeta^2(v_t + g(u)v_x + f(v)v_y). \tag{11}$$

$$\sigma_t + \sigma_u u_t + \sigma_v v_t + g(u)(\sigma_x + \sigma_u u_x + \sigma_v v_x) + f(v)(\sigma_y + \sigma_u u_y + \sigma_v v_y) + \sigma g'(u)u_x - \varsigma f'(v)u_y$$
$$= \zeta^3(u_t + g(u)u_x + f(v)u_y) + \zeta^4(v_t + g(u)v_x + f(v)v_y). \tag{12}$$

The comparison of the coefficients of $u_t$ and $v_t$ on both sides of Eq. (11) yields $\zeta^1 = \varsigma_u$ and $\zeta^2 = \varsigma_v$ respectively. These facts reduce Eq. (11) to

$$\varsigma_t + g(u)\varsigma_x + f(v)\varsigma_y + \varsigma f'(v)v_y - \sigma g'(u)v_x = 0. \tag{13}$$

Similar arguments reduce Eq. (12) to

$$\sigma_t + g(u)\sigma_x + f(v)\sigma_y + \sigma g'(u)u_x - \varsigma f'(v)u_y = 0. \tag{14}$$

Since both $\varsigma$ and $\sigma$ are neither functions of $u_i$ nor $v_i$ $(i = x, y)$ coupled with the fact that $g(u)$ and $f(v)$ have non-vanishing first derivatives, we can see that Eqs. (13) and (14) can be satisfied if and only if $\varsigma = \sigma = 0$. Thus the system of two-dimensional Burgers Eq. (1) is not non-linearly self-adjoint in the sense of Definition 2.3.

According to [7], substitution (9) can be replaced with a more general substitution where $\varsigma$ and $\sigma$ involve not only the independent variables but also dependent variables and their derivatives. That is, the system of two-dimensional Burgers Eq. (1) will be non-linearly self-adjoint if there exists differential substitutions

$$\alpha = \varsigma(x,y,t,u,v,u_k,v_k,u_{kk},v_{kk},\ldots), \quad \varsigma \neq 0$$
$$\beta = \sigma(x,y,t,u,v,u_k,v_k,u_{kk},v_{kk},\ldots), \quad \sigma \neq 0 \tag{15}$$

involving a finite number of partial derivatives of $u$ and $v$ with respect to $k$ (independent variables), such that the equation

$$F^*\big|_{\substack{\alpha = \varsigma \\ \beta = \sigma}} = \zeta_0 F + \zeta_1 D_k(F) + \zeta_2 D_{kk}(F) + \cdots \tag{16}$$

holds identically in the variables $x, y, t, u, v, u_k, v_k, \ldots,$ where $\zeta_0, \zeta_1, \ldots$ are undetermined variable coefficients different from $\infty$ on the solutions of Eq. (1).

Taking into account Eqs. (5) and (16), we obtain the following result.

**Theorem 2.1.** *The system of two-dimensional Burgers Eq. (1) is non-linearly self-adjoint if*

$$\alpha = v_k,$$
$$\beta = -u_k, \tag{17}$$

*for $k = x, y, t$. These substitutions satisfied (16) for*

$$\zeta_i = \begin{cases} 0, & i \neq 1 \\ 1, & i = 1 \end{cases}$$

[1] The proof is similar to the one given for non-linearly self-adjoint with point-wise substitution.