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Unsteady axisymmetric boundary-layer equations: Transformations, properties, exact solutions, order reduction and solution method

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ABSTRACT

The paper deals with equations describing the unsteady axisymmetric boundary layer on a body of revolution. The shape of the body is assumed to be arbitrary. The axisymmetric boundary-layer equation for the stream function is shown to reduce to a plane boundary-layer equation with a streamwise-coordinate-dependent viscosity of the form

$$w_{tz}+w_z w_{xz}-w_x w_{zz}=\nu r^2(x) w_{zzz}+F(t,x).$$

We describe a number of new generalized and functional separable solutions to this non-linear equation, which depend on two to five arbitrary functions. The solutions are obtained with a new method (direct method of functional separation of variables) based on using particular solutions to an auxiliary ODE. Many of the solutions are expressed in terms of elementary functions, provided that the arbitrary functions are also elementary. Two theorems are stated that enable one to generalize exact solutions of unsteady axisymmetric boundary-layer equations by including additional arbitrary functions. Furthermore, we specify a von Mises-type transformation that reduces the unsteady axisymmetric boundary-layer equation to a non-linear second-order PDE. We also present several new exact solutions to the plane boundary-layer equation and solve a boundary layer problem for a non-uniformly heated flat plate in a unidirectional fluid flow with temperature dependent viscosity.

The method proposed in this paper can also be effective for constructing exact solutions to many other non-linear PDEs.

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1. Introduction. The classes of equations considered

1.1. Preliminary remarks

Hydrodynamic boundary-layer equations are important and fairly common in various areas of science and engineering (e.g., see [1–4]).

Exact solutions to the Navier–Stokes, boundary-layer, and related equations contribute to better understanding of qualitative features of steady and unsteady fluid flows at large Reynolds numbers; these features include stability, non-uniqueness, spatial localization, blow-up regimes, and others.

Exact solutions with significant functional arbitrariness are of particular interest because they may be used as test problems to ensure efficient estimates of the domain of applicability and accuracy of numeric, asymptotic, and approximate analytical methods for solving suitable non-linear hydrodynamic-type PDEs as well as certain model problems.

1.2. Plane boundary-layer equations

The system of unsteady plane laminar boundary-layer equations is written as [1,2]

$$U_t+UU_x+VU_y=\nu U_{yy}+F(t,x), \tag{1}$$

$$U_x+V_y=0 \tag{2}$$

where t is time, x and y are longitudinal (streamwise) and transverse coordinates (tangential and normal to the body surface), U and

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V are the longitudinal and transverse fluid velocity components, $F(t,x)=-p_x/\rho$ is a given function, p is the pressure, ρ is the mass density, and ν is the kinematic viscosity of the fluid. The fluid is assumed to be incompressible.

With the stream function W defined by

$$U=W_y, \quad V=-W_x, \tag{3}$$

system (1)–(2) reduces to a single non-linear third-order equation [1,2]:

$$W_{ty}+W_yW_{xy}-W_xW_{yy}=\nu W_{yyy}+F(t,x). \tag{4}$$

Exact solutions and transformations of Eq. (4) as well as different problems on the hydrodynamic boundary layer have been addressed by many researchers (e.g., see [1–23]). For invariant and non-invariant exact solutions to this equation in the steady case (with $W_t=0$), see the papers [1–3,5,8,10,12,14,17,20,21,24]. Some exact solutions and transformations of the unsteady plane boundary-layer equation (4) can be found in [6,7,9,11,13,15,16,18–20,23].

Remark 1. The studies [17,20,24] present some exact solutions to an axisymmetric boundary-layer equation on an extensive body of revolution; the equation dealt with in these studies can formally be obtained from Eq. (4) by substituting $(yW_{yy})_y$ for W_{yyy}.

1.3. Axisymmetric unsteady laminar boundary-layer equations on a body of an arbitrary shape

The system of axisymmetric unsteady laminar boundary-layer equations is [1,3]

$$U_t+UU_x+VU_y=\nu U_{yy}+F(t,x), \tag{5}$$

$$(rU)_x+(rV)_y=0, \quad r=r(x), \tag{6}$$

where x and y are the longitudinal (streamwise) and transverse coordinates (with the respective unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ being tangential and normal to the surface of the body of revolution), U and V are the streamwise and transverse fluid velocity components, $F(t,x)=-p_x/\rho$ is a given function, and $r=r(x)$ is the dimensionless cross-sectional radius perpendicular to the axis of rotation and defining the shape of the body. The other notations are the same as in Eq. (4). The choice of the unit of length to define $r=r(x)$ does not affect the form of the above equations.

Introducing a stream function W such that

$$U=W_y, \quad V=-W_x-\frac{r'_x}{r}W, \quad r=r(x), \tag{7}$$

reduces system (5)–(6) to a single third-order equation [1,3]:

$$W_{ty}+W_yW_{xy}-W_xW_{yy}-[\ln r(x)]'_x WW_{yy}=\nu W_{yyy}+F(t,x). \tag{8}$$

If $r(x)=\text{const}$, Eq. (8) coincides with (4). If $r(x)\neq\text{const}$, Eq. (8) becomes much more complicated to analyze. For some specific $r(x)$, it was shown in [25] that Eq. (8) with three independent variables can be reduced to an ordinary differential equation or a partial differential equation with two independent variables; the analysis was based on a modification of the Clarkson–Kruskal direct method [20,26,27].

In what follows, we deal with the general case of Eq. (8) with arbitrary $r(x)$. Allowable forms of the pressure function $F(t,x)$ will, as usual, be determined during the analysis.

2. A transformation of the equation. Generalized and functional separable solutions

2.1. Reduction to an unsteady plane boundary-layer equation with varying viscosity

By introducing the new variables

$$z=r(x)y, \quad w=r(x)W, \tag{9}$$

we rewrite Eq. (8) in the form

$$w_{tz}+w_zw_{xz}-w_xw_{zz}=\nu r^2(x)w_{zzz}+F(t,x), \tag{10}$$

which can be treated as an unsteady plane boundary-layer equation with varying viscosity,

$$\nu_e=\nu r^2(x),$$

dependent on the streamwise coordinate x. The pressure function $F(t,x)$ remains unchanged.

In the special case $r(x)\equiv 1$, Eq. (10) coincides with the plane boundary-layer equation (4).

A physical interpretation of a boundary-layer equation with varying viscosity $\nu_e=\nu_e(x)$, unrelated to system (5)–(6), is given below in Section 9.

2.2. Generalized and functional separable solutions

In subsequent sections, we will be dealing with Eq. (10) and present a number of its generalized separable solutions of the form

$$w=\sum_{i=1}^{n} f_i(t,x)u_i(z). \tag{11}$$

The functions $f_i(t,x)$ and $u_i(z)$ are determined during the analysis of the equation resulting from inserting (11) into (10). Solution (11) most frequently involves the following functions:

$$u_i(z)=z^m \ (m=0,\ 1,\ 2), \quad u_i(z)=\exp(\lambda_i z),$$
$$u_i(z)=\cos(\beta_i z), \quad u_i(z)=\sin(\beta_i z),$$

where λ_i and β_i are unknown parameters.

Remark 2. The books [20,28,29] detail various modifications of the method of generalized separation of variables based on seeking solutions of the form (11). These books give a large number of non-linear PDEs and systems of PDEs admitting generalized separation of variables.

Apart from generalized separable solutions, this paper will present more complex functional separable solutions of the form

$$w=\sum_{i=1}^{n} f_i(t,x)u_i(\xi), \quad \xi=\varphi(t,x)z+\psi(t,x). \tag{12}$$

The functions $f_i(t,x)$, $\varphi(t,x)$, $\psi(t,x)$, and $u_i(\xi)$ are determined during the analysis of the equation resulting from inserting (12) into (10).

Remark 3. A number of exact solutions of the form (11) and (12) to unsteady plane boundary-layer equations (4) can be found in [11,13,15,20,28]. For steady and unsteady generalized and functional separable solutions to two- and three-dimensional Navier–Stokes equations, as well as some other exact solutions, see [19,20,23,30–38]. For models and exact solutions of hyperbolic and differential-difference Navier–Stokes equations, see [39–42].

The generalized and functional separable solutions to non-linear diffusion-type equations can be found in [43–47] (see also [20,29] and references therein).

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