International Journal of Non-Linear Mechanics 68 (2015) 87–95

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Magic angles for fiber reinforcement in rubber-elastic tubes subject to pressure and swelling

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ARTICLE INFO

Article history:
Received 25 March 2014
Accepted 7 May 2014
Available online 4 June 2014

Keywords:
Swelling
Rubber elasticity
Fiber reinforcement
Anisotropy
Finite strain

ABSTRACT

The synthetic fibers in a fire hose are specially oriented so as to minimize the jerk when the hose is suddenly pressurized – at other fiber orientations the hose would either extend abruptly or shorten abruptly. This is one example of a magic angle that separates different response directions in a deformation mode as some driving mechanism is varied. Here we investigate magic angles for winding fibers around a tube with an otherwise rubbery matrix by providing a hyperelastic analysis. While the driving mechanism of pressurization is well known in this regard, we also consider the driving mechanism of matrix swelling. Magic angles of fiber orientation are considered for axial deformation and for channel constriction.

1. Introduction

A rubber tube that is reinforced with helically wound fibers and then subject to internal pressure may either increase or decrease its length. Similarly, it may either decrease or increase its inner channel radius. For sufficiently thin tubes with sufficiently stiff fibers the effects are correlated: length decrease accompanies radius increase, length increase accompanies radius decrease. The former occurs if the fibers are close to an axial orientation, the latter if the fibers are close to a circumferential orientation. The special "magic" winding angle (with respect to the axial direction) of $\arctan\sqrt{2} = \arccos 1/\sqrt{3} = 54.7^\circ$ provides the transition between these responses in thin tubes with sufficiently stiff fibers [9]. Fibers at this angle are naturally aligned to support both the axial stress and the hoop stress that is generated by the pressurization. Fibers aligned at any other angle must be rotated by the deformation so as to achieve a stress-supporting alignment.[1]

It is likely that the spray hose on your kitchen sink displays fiber reinforcement that is close to this angle. The synthetic fibers in fire hose are aligned at this angle so as to minimize jerk when the hose is suddenly turned on.

If the matrix is capable of carrying load then the magic angle will generally shift. The linear elastic treatment for determining this shift in thin tubes follows from what is known as netting analysis [5]. Accounting for a finite wall thickness and allowing non-linear elastic properties provide for additional changes in the special magic angle. It now also becomes possible for a length increase to accompany a radius increase [8]. These observations hold great interest not only for structural applications, but also for the design of actuators that can serve as artificial muscles [10] and for explaining deformation and motion in a variety of biological contexts [8]. In particular, the analysis of [8] provides a systematic means for exploring paths in design space so as to find locations where there is a qualitative change in some specific deformation characteristic, e.g., the magic angle of fiber winding where axial lengthening gives way to axial shortening. This analysis applies also to paths in load space (at fixed design) so as to, for example, determine specific values of finite pressure that separate axial lengthening from axial shortening.

Similar transitions between different qualitative behaviors in the mechanical deformation response of fiber reinforced rubbery tubes have been noted in [2,1,3], again in the context of finite thickness tubes with non-linear elastic behavior. The difference is that it is not channel pressurization that causes the deformation in these studies, rather it is an overall material *swelling* that drives the deformation. In [2,1,3], swelling causes the matrix – the medium in which the fibers are embedded – to expand. However

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[1] The magic angle value $\beta = 54.7^\circ$ follows from free body analysis of a sufficiently small (and hence sufficiently flat) triangular membrane element. One edge surface has normal in the axial direction and carries a normal axial stress. Another edge surface has normal in the circumferential direction and carries hoop stress. The hypotenuse is aligned in the fiber direction and so is traction free (take a Cauchy cut through the non-loadbearing matrix). Eliminating the fiber load between the two force balance equations then gives $\tan^2\beta = 2$. Here 2 is the ratio of hoop stress to axial stress in a thin walled pressure vessel. One factor of $\tan\beta$ comes from the resolution of the fiber loading vector into component directions. The other $\tan\beta$ is the area ratio of surface carrying axial stress to surface carrying hoop stress.

http://dx.doi.org/10.1016/j.ijnonlinmec.2014.05.008
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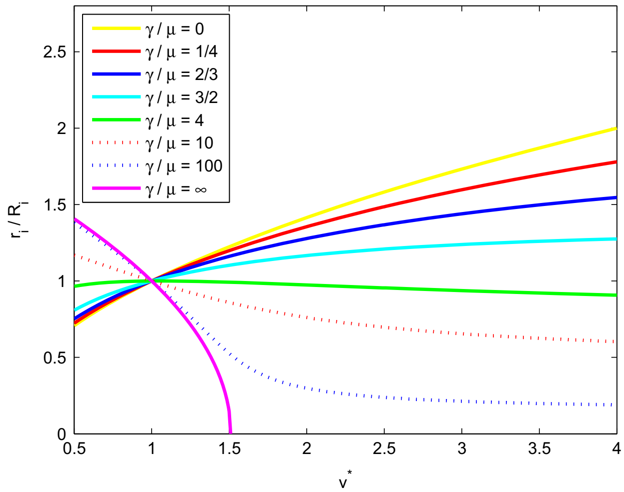

Fig. 1. Deformed location of tube channel radius r_i as a function of swelling $\overset{*}{v}$ ($\overset{*}{v}=1$ is the reference state of no swelling) for an unpressurized thick tube with initial inner radius R_i, initial outer radius R_o, and $R_o=4R_i$. Different curves correspond to tubes with different fiber/matrix stiffness ratios with γ and μ defined according to (8), (42), (43), and fiber angle $\beta=\pi/6$ (rad) $=30^\circ$ with respect to the tube axis. Solid curves correspond to the six curves depicted in Figure 4 of [2].

the fibers themselves do not expand and so become stretched as the matrix expands. Hence swelling leads to fibers in tension within a matrix that is in compression. The resulting deformation, which would be a simple homogeneous volume expansion in the absence of fibers, becomes a more complicated inhomogeneous expansion due to the fiber action. For helically wound fibers this would in general lead to a torsional deformation as studied in [3] and later observed in the controlled experiment of [7]. The torsion can be eliminated by making a symmetrical fiber winding (clockwise-counterclockwise balance), but in that case there is still an inhomogeneous radial expansion as described in [2].

Fig. 1 is adapted from [2] and shows how the channel radius of a tube can change due to swelling. In this example there is no pressurization, and a single winding angle is considered. The different curves correspond to different ratios of a fiber stiffness parameter to the matrix stiffness parameter. The interesting aspect is that, while the outer radius always expands (not shown), the inner radius could either expand or contract. Fibers that exhibit relatively low stiffness (γ/μ small in Fig. 1) lead to expansion of the inner radius – as would be the case under homogeneous expansion when no fibers are present. In contrast, fibers that are relatively stiff compared to the matrix (γ/μ large) lead to channel constriction as the matrix expands. For the example depicted in Fig. 1 it follows that if the volumetric expansion exceeds about 1.5 times the original volume then the channel opening can be made arbitrarily small by taking a sufficiently high fiber/matrix stiffness quotient. This is a finite deformation effect; the linear theory, even when generalized to include volume change,[2] is ill formulated for analyzing full channel closure.

Because Fig. 1 shows both channel opening and channel closing under swelling as determined by the relative stiffness of the fibers to the matrix, it follows that there will be certain stiffness ratios at which the channel maintains a relatively constant opening as the swelling proceeds. This is also apparent from Fig. 1 which shows that certain curves are very close to horizontal. In view of the finite nature of the deformation, one can characterize such an effect either globally (e.g., small variation in radius over some finite swelling range of interest) or locally (curve locations with horizontal tangency). Fig. 1, while showing certain relatively flat curves, does not show any specific locations of horizontal tangency in the depicted range of swelling volume increase ($1<\overset{*}{v}<4$). However other examples (including the next figure) do show such locations.

Fig. 2, also adapted from [2], shows an additional aspect of such tube swelling when a constant pressurization is present. Each curve now corresponds to the same tube characteristics but subject to a different constant pressurization as the swelling proceeds. For relatively lower pressure the swelling expands the channel. For relatively higher pressure the swelling constricts the channel. The overall tube length is held fixed in this example, and one may then calculate the axial force needed to maintain the fixed length. Sometimes this axial force is tensile while in other cases the axial force is compressive. Fig. 2 curve locations marked with a star correspond to zero axial force.[3]

The purpose of this paper is to draw out these connections as they relate to the direction of the fiber reinforcement. Specifically we consider both pressure and swelling as forcing variables. The basic constitutive theory for a fiber reinforced swellable hyperelastic material is presented in Section 2 where the stored energy density is taken in the form $\Phi_m(I_1,\overset{*}{v})+\Phi_f(I_4)$. Here Φ_m and Φ_f are separate matrix and fiber contributions, I_1 and I_4 are the usual invariants, and $\overset{*}{v}$ is the swelling ratio. This is followed in Section 3 with the formulation and solution of the relevant boundary value problem for a tube in terms of a general Φ_m and Φ_f. Specific forms for Φ_m and Φ_f are taken in Section 4 so as to illustrate the possibilities for channel opening vs. channel constriction, as well as for axial lengthening vs. axial shortening, as the swelling and pressure are varied. Among other things this shows how different behaviors can ensue as a function of fiber winding angle, all other structural and material aspects of the tube being held fixed. A methodology for determining magic angles of fiber winding as a function of pressurization (at fixed swelling) is presented in Section 5. Separate criteria are needed for channel opening and for axial lengthening in the finite deformation theory of thick walled tubes with non-trivial matrix stiffness. If *all* of the following limits hold simultaneously: small deformation, thin wall, and no matrix stiffness, then these two separate criteria give the same result and retrieve the classical result of 54.7°. A complementary analysis for determining magic angles under swelling (at fixed pressurization) is presented in Section 6. Section 7 provides concluding remarks.

2. General framework

Let $\mathbf{X}$ be a generic position vector in a reference configuration $\Omega_{\mathbf{X}}$ that is regarded as the state of an unloaded body prior to swelling. The load is described in the standard way in terms of boundary tractions and body forces. Together, swelling and the application of load give rise to an invertible deformation $\mathbf{x}(\mathbf{X})$ that maps $\Omega_{\mathbf{X}}$ to the configuration $\Omega_{\mathbf{x}}$. The deformation gradient is $\mathbf{F}=\partial\mathbf{x}/\partial\mathbf{X}$. Both the compressible theory of hyperelasticity and the incompressible theory of hyperelasticity can be generalized so as to include a notion of swelling [15]. We consider the generalization of the incompressible theory. This means that the volume is still prescribed, but the prescribed value of the volume is greater than one in the event that swelling occurs. Thus there is a local

[2] That is, anisotropic linear elasticity with eigenstrains.

[3] Qualitative changes are described in [8] in terms of inversion points and perversion points on a load path. In this terminology, Fig. 2 locations of horizontal tangency correspond to an inversion point of the channel radius deformation parameter. With respect to the unconstrained and unswollen tube, star locations in Fig. 2 can be viewed as corresponding to perversion points in the constraining force.