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A linear complementarity problem formulation for periodic solutions to unilateral contact problems

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ABSTRACT

Presented is an approach for finding periodic responses of structural systems subject to unilateral contact conditions. No other non-linear terms, e.g. large displacements or strains, hyper-elasticity, plasticity, etc. are considered. The excitation period due to various forcing conditions—from harmonic external or contact forcing due to a moving contact interface—is discretized in time, such that the quantities of interest—displacement, velocity, acceleration as well as contact force—can be approximated through time-domain schemes such as backward difference, Galerkin, and Fourier. The solution is assumed to exist and is defined on a circle with circumference T to directly enforce its periodicity. The strategy for approximating time derivative terms within the discretized period, i.e. velocity and acceleration, is hence circulant in nature. This results in a global circulant algebraic system of equations with inequalities that can be translated into a unique Linear Complementarity Problem (LCP). The LCP is then solved by dedicated and established methods such as Lemke's Algorithm. This allows for the computation of approximate periodic solutions exactly satisfying unilateral contact constraints on a discrete time set. The implementation efficiency and accuracy are discussed in comparison to classical time marching techniques for initial value problems combined with a Lagrange Multiplier contact treatment. The LCP algorithm is validated using a simple academic problem. The extension to large-scale systems is made possible through the implementation of a Craig–Bampton type Modal Component Synthesis. The method shows applicability to industrial rotor/casing contact set-ups as shown by studying a compressor blade. A good agreement to time marching simulations is found, suggesting a viable alternative to time marching or Fourier-based harmonic balance simulations.

1. Introduction

The study of periodic solutions of harmonically excited systems subject to unilateral contact constraints is of interest for many applications, such as predicting brake disk vibrations [1], stability analysis of delayed systems subject to material removal [2], study of limit cycles in Lur's feedback control systems for non-smooth mechanical systems [3,4] or mesh stiffness variation in gears-pairs [5–7]. Popularly, methods such as harmonic balance based [8] or shooting methods [9] are employed to find solutions of smooth non-linear systems in an efficient manner. Yet, both of these approaches face some inherent downfalls when non-smooth non-linearities—systems exhibiting non-differentiability of discontinuities in the unknowns—are encountered. On the one hand, the Harmonic Balance Method (HBM) is known to produce poor approximations of non-smooth functions with a finite number of harmonics, producing artefacts such as the Gibbs phenomenon [10]. Hence penalty-like approximations of the contact inequalities are introduced [1] to effectively smoothen the non-smoothness. On the other hand, shooting methods in a contact framework can face ill-conditioned gradients, making convergence difficult and increasing computational times. Furthermore, reaching a purely steady state solution within a heavy-duty time-marching simulation may be impossible to achieve due to the high sensitivity of the solution with respect to system parameters, such as stiffness, frequency, and gap.

The so-called Linear Complementarity Problem (LCP) deals with a linear algebraic system of equations subject to inequality constraints. Originally presented in [11–13] as an alternative formulation for quadratic programming, a few notable solution methodologies have been introduced: the classical pivoting, also generally referred to Lemke's algorithm [11], and iterative methods such as the Gauss–Seidel method [14,15] and Newton-like methods [16] are among the most popular. With applications in many scientific fields, the LCP has found a strong foothold especially in economic engineering, mathematical programming, games theory and recently switched electronic systems [3,17,18]. The idea is that LCP solvers can find solutions of underlying linear

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Nomenclature

Roman symbols

$\mathbf{A}$	LCP coefficient matrix
$\mathbf{B}$	contact constraints matrix
$\mathbf{I}$	identity matrix
$\mathbf{L}$	linear operator
$\mathbf{M}$,$\mathbf{C}$, $\mathbf{K}$	mass, damping and stiffness matrices
$\mathfrak{F}$	discreet Fourier transform
$\ddot{\mathbf{x}}$,$\mathbf{a}$	acceleration vectors
Δt	time step
$\dot{\mathbf{x}}$,$\mathbf{v}$	velocity vectors
$\mathbf{d}$	reference wall position
$\mathbf{f}$	external forcing vector
$\mathbf{g}(\mathbf{x})$	gap function
$\mathbf{q}$	underlying linear system solution
$\mathbf{w}$,$\mathbf{z}$	complementary vectors
$\mathbf{x}$	displacement vector
$\mathbf{x}_\mathrm{b}$, $\mathbf{x}_\mathrm{i}$, $\mathbf{x}_\mathrm{c}$	boundary, internal and contact DOF
$\mathbf{x}_\mathrm{cb}$	Craig–Bampton DOF
a, b	Fourier coefficients
k	number of time steps
m	number of contact constraints
n	number of DOF
p	number of Fourier terms
T	excitation period
t	time

Greek symbols

$\mathbf{\Lambda}$	eigenvalue matrix
$\boldsymbol{\lambda}$	contact force vector
$\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{\Theta}$	weighting functions
$\mathbf{\Phi}_\mathrm{cb}$	Craig–Bampton reduction matrix
$\mathbf{\Phi}_\mathrm{i}$, $\mathbf{\Phi}_\mathrm{s}$	internal and static eigenvectors
ω	excitation frequency
ω_0	first natural frequency
ϕ_i	shape function

Abbreviations

BDS	backward difference scheme
CDS	central difference scheme
DOF	degree of freedom
EO	engine order
EOM	equation of motion
ETM	explicit time-marching
FEDT	finite element discretization in time
HBM	Harmonic Balance Method
HDHBM	High Dimension Harmonic Balance Method
LCP	linear complementarity problem
LE	leading edge
MC	mid-chord
TE	trailing edge

algebraic systems subject to non-linear Kuhn–Tucker-like conditions. The LCP method has already been applied to piecewise linear mass–spring systems in a time-marching framework, solving an LCP for every time step [19]. In a similar fashion, the application of LCPs to transient analysis frictional problems is reported in [20].

This paper presents an approach for finding periodic solutions to systems of ODEs involving inequalities within a unique LCP. Firstly, the general LCP formulation is outlined. Next, different time-derivative approximations are detailed for constructing the LCP system. An academic application is presented, specifically looking at a comparison of the LCP results to classic heavy-duty time-marching simulations. Finally, the application of the LCP method to an industrial compressor-blade geometry is explored with a detailed focus on frequency domain analysis of the responses.

2. Linear complementarity problem

The formulation of an LCP lends itself explicitly to treat linear mechanical systems subject to Kuhn–Tucker conditions such as unilateral contact. Forced and possibly large-scale mechanical systems undergoing unilateral contact conditions that are T-periodic are targeted, see Fig. 1. A periodic displacement of the discretized model in space $\mathbf{x}(t)$ of period T in time is assumed to exist. Accordingly, it is a solution to the following combined equations and inequations:

$$\forall t \in \mathbb{S}_T, \quad \begin{cases} \text{equation of motion} & \text{(a)} \\ \mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)+\mathbf{B}^\top\boldsymbol{\lambda}(t)=\mathbf{f}(t) & \\ \text{complementarity conditions} & \text{(b)} \\ \boldsymbol{\lambda}(t)\le\mathbf{0}, \quad \mathbf{g}(\mathbf{x}(t))\le\mathbf{0} \quad \text{and} \quad \boldsymbol{\lambda}(t)^\top\mathbf{g}(\mathbf{x}(t))=0 & \end{cases} \tag{1}$$

where $\mathbb{S}_T$ is the circle of circumference T on which the periodic solution is defined. The complementarity conditions can be read thus: the contact force $\boldsymbol{\lambda}(t)$ may only take on values that ensure a compressive force. The gap $\mathbf{g}(\mathbf{x}(t))$ may only take on values ensuring impenetrability. The product of both must be zero ensuring that contact forces may only exist if the gap is nil and vice versa.[1] The equation of motion (EOM) in Eqs. (1a) and (1b) is derived from a discretized finite-element model with n degrees of freedom (DOF) and m contact conditions (for which generally $m \ll n$), where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K} \in \mathbb{R}^{n\times n}$ are the mass, damping and stiffness matrices respectively. The harmonic external forcing $\mathbf{f}(t) \in \mathbb{R}^n$ is of period T and displacements $\mathbf{x}(t) \in \mathbb{R}^n$ are mapped to the contact force term $\boldsymbol{\lambda}(t) \in \mathbb{R}^m$ by the contact constraint matrix $\mathbf{B} \in \mathbb{R}^{m\times n}$ through a linear gap function:

$$\mathbf{g}(\mathbf{x}(t)) = -\mathbf{B}\mathbf{x}(t)-\mathbf{d}(t) \tag{2}$$

where $\mathbf{d}(t)$ is the reference position of a potentially time-dependent moving rigid wall. The impenetrability condition is expressed by inequality constraints on the gap function $\mathbf{g}(\mathbf{x}(t))$ and contact force $\boldsymbol{\lambda}(t)$ as well as the complementarity criteria in Eq. (1b). It should be noted here that the gap function is considered to be purely linear in $\mathbf{x}(t)$ as expressed by Eq. (2). Systems for which the constraint matrix varies within the period of interest, e.g. large tangential displacements or strongly deformation dependent contact interfaces, cannot be accounted for within this approach. In order to simplify the computation, contact between a flexible body and a rigid wall is assumed, although flexible multibody contact problems do not pose a mathematical limitation and the equations explicitly remain the same.

To transform Eqs. (1a) and (1b) into an equivalent LCP able to capture periodic solutions, for a given period T, a few steps are necessary. Firstly, a time-discretization is needed. Assuming an

[1] In the notation used throughout this documentation, an open gap and a compressive force are assumed to be $\mathbf{g}(\mathbf{x}(t)) \le 0$ and $\boldsymbol{\lambda} \le 0$ respectively. Other sign conventions are possible.