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A non-linear one-dimensional model of cross-deformable tubular beam

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ABSTRACT

A direct non-linear one-dimensional model of an elastic, thin-walled, planar beam is formulated. The model accounts for changes in shape of the cross-section, in particular the ovalization (or flattening) occurring in tubular beams. The deformation of the cross-section is described in the spirit of the Generalized Beam Theory, as a linear combination of known deformation modes and unknown amplitude functions, said to be distortions. Kinematics calls for introducing distortional and bi-distortional strains, in addition to the usual strain measures of rigid cross-section beams. The balance equations are derived through the Virtual Power Principle, in which distortional and bi-distortional stresses, as well as distortional forces, are defined as conjugate quantities of distortional strain-rates and velocities, respectively. A non-linear, fully coupled, hyperelastic law is assumed. All the distortional quantities and the constitutive law are identified, via energy equalities, from a three-dimensional fiber-model of thin-walled beam where, for simplicity, just a distortion mode is considered. The model is specialized to a Euler–Bernoulli tubular beam, in which only constitutive non-linearities are retained, while kinematics is linearized. The relevant non-linear equations are solved, via a perturbation method, for several static loadings and for large-amplitude free vibrations. The interaction occurring between global bending and cross-section distortion is analyzed.

1. Introduction

Thin-walled beams (TWB) suffer changes of the shape of their cross-sections, both in-plane (distortion) and out-of-plane (warping). These deformations, different from what happens in compact cross-sections, are *not* secondary kinematic effects, but they strongly affect the mechanical performances of TWB. Well-known examples of such distortional mechanisms are the following: (a) *warping* caused by the non-uniform torsion of open TWB, which leads to a remarkable increment of the torsional stiffness, compared to that of de Saint-Venant [1]; (b) *distortional buckling*, occurring in compressed open or closed TWB, in which the walls of the beam bend in the cross-section plane, causing dangerous effects when the mode interacts with Eulerian or flexural–torsional modes (see, e.g. [2,3]); and (c) *ovalization* (or flattening), manifesting in bent tubular beams, which reduces the bending stiffness, possibly causing the collapse of the structure even in the elastic phase [4].

Warping of cross-undeformable beams has received great attention in the literature [5–13], where several models were formulated, aimed to extend the Vlasov linear theory to the non-linear field. In these papers, the key-point of the theory is how to link the warping to the twist, in order to keep the model one-dimensional. When the model is derived from a 3D continuum, the task is accomplished by invoking the classical Vlasov hypotheses (inextensibility of the middle-line and shear-undeformability of the middle-surface) [7]; on the other hand, when the model is direct (one-dimensional structured continuum), the same goal is reached by introducing suitable internal constraints among the kinematic descriptors [8–10].

In-plane distortion has also been extensively studied, mostly in buckling, but often confined to the linear field. The approach followed is either purely numerical [14–17] or semi-analytical (Generalized Beam Theory (GBT) [18–21]). Few examples of semi-analytical non-linear models are known to the authors [18]; in particular the variational-asymptotic method (VAM) has been exploited in Ref. [44] to catch the Brazier effect in circular tubes. In some of these papers, the TWB is considered as an assembly of plates, where the displacements on the cross-section are either interpolated between nodal values (in finite strips) or described as linear combinations of assumed modes (in GBT). The result is a one-dimensional model depending on the nodal displacements or modal amplitudes, all functions of the axis-abscissa (and on time).

Derivation from a 3D model, however, is not straightforward, when geometrical non-linearities have to be accounted for, since the interpolation functions or the distortional modes should be

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carefully chosen in order to avoid locking problems [22] due to the fact that rigid motions cannot always be exactly described. In this perspective, a direct approach, like that used in [8,9] for warping only, seems more appealing, since kinematics is exact in that context, and no interpolation of the displacement field is required. On the other hand, the direct approach has the drawback that the constitutive law has to be postulated, so that often the authors have to borrow it from the literature of the 3D-models (see [9,23]).

A direct approach to the modeling of the rod bending is presented in [24–26] for the case of non-homogeneous beams, where links between direct 1D statements and formulations of the 3D elasticity are also established.

In some cases, models are formulated after the identification of the micro-kinematics starting from macro-kinematics, when both macro- and micro-models are three-dimensional, as done for instance in [27–30].

This paper is an attempt to formulate a 1D direct model of planar beam undergoing in-plane distortion of the cross-section, with a constitutive law identified by a *consistent* 3D model. Here, consistency refers to the fact that the kinematics of each fiber of the 3D model is described by the same kinematics of the 1D model. To this purpose, the fundamental idea of the GBT is borrowed, i.e. the distortion is described by assumed distortional modes. Rigid motions are exactly described when the distortional amplitudes are zero. The model presented here is generally formulated under a large displacement regime (in contrast with most of the papers focused on GBT), taking into account both (un-constrained) warping and flattening of the cross-section, described by a generic number of shape functions. Then, it is specialized to a tubular beam to investigate, under simplifying assumptions, the ovalization effects.

In the model presented here, some degrees of freedom describing the deformation of cross-section are coupled to the overall displacement of the beam. This occurrence could lead to the design of an optimally shaped beam, able to trap energy in its internal vibration modes which, in the considered case, are the section vibration modes, as done, e.g. in [31–34]. Moreover the model can be suitably used, for instance, for applications related to soft impact of cantilevers [35–38].

The paper is organized as follows. In Section 2 a direct 1D planar model of TWB with double-symmetric and deformable cross-section is formulated. In Section 3 a fiber model is introduced, and all the quantities of the 1D model are identified for the simplest case of a unique distortional mode. In Section 4 the model is specialized to a tubular beam. In Section 5 several static and dynamic loadings are considered for some paradigmatic examples where flattening is involved. Relevant problems are solved via a perturbation approach; moreover, numerical results are displayed. In Section 6, some conclusions are drawn. In Appendix A some details of computations are shown.

2. A direct 1D model of TWB with deformable cross-section

Let us consider a beam as a one-dimensional polar continuum, endowed with additional kinematic descriptors, able to (roughly) describe the loss of shape of the cross-section. We confine ourselves to double-symmetric cross-sections for which the centroid axis coincides with the flexural center-axis. Denoting $\overline{\mathbf{x}}_G(s)$ the position of the beam axis in the reference configuration, where $s \in [0, l]$ is a curvilinear abscissa, and $\mathbf{x}_G(s,t)$ its current position at the time t, the displacement field is described by the translation $\mathbf{u}_G(s,t) := \mathbf{x}_G - \overline{\mathbf{x}}_G$, and by the rotation $\mathbf{R}(s,t)$, which leads the principal inertia triad $\overline{\mathcal{B}} := (\overline{\mathbf{a}}_1, \overline{\mathbf{a}}_2, \overline{\mathbf{a}}_3)$ of directors $\overline{\mathbf{a}}_i$, $i = 1,2,3$ attached to the body-point P in the reference configuration to match the triad $\mathcal{B} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ of the same directors in the current configuration. Moreover, a set of n scalar fields $(a_1(s,t), a_2(s,t), \ldots a_n(s,t))$ to be referred as *distortional variables* is introduced. Strains are defined as follows:

$$\begin{aligned} &\mathbf{e}_G := \mathbf{R}^T(\overline{\mathbf{a}}_1 + \mathbf{u}_G') - \overline{\mathbf{a}}_1 \\ &\mathbf{k} := \text{axial}[\mathbf{R}^T\mathbf{R}'] \\ &\alpha_j := a_j \\ &\beta_j := a_j' \end{aligned} \tag{1}$$

in which a prime denotes s-differentiation and axial[·] stands for the axial vector of the skew symmetric tensor in argument. Here, $\mathbf{e}_G$ and $\mathbf{k}$ are, respectively, the strain-vector and curvature-vector usually adopted for locally rigid beams (i.e. beam with rigid cross-sections, see, e.g. [39]), while α_j, β_j are additional strains, peculiar to locally deformable beams. Various other vectorial parametrizations of rotation and related strain measures, some of them are different from Eq. (1b), are discussed in [40]. Here α_j will be referred to as the *distortional strains* and β_j as the *bi-distortional strains* (or *distortion-gradients*). It should be noticed that *kinematics of the rigid and deformable cross-section is uncoupled*, since $\mathbf{e}$ and $\mathbf{k}$ only depend on $\mathbf{R}$ and $\mathbf{u}_G$, while α_j and β_j only depend on a_j.

The geometric boundary conditions prescribe the values of the configuration variables at the ends, if a constraint is applied, namely

$$\mathbf{u}_{GH} = \hat{\mathbf{u}}_{GH}(t), \quad \mathbf{R}_H = \hat{\mathbf{R}}_H(t), \quad a_{jH} = \hat{a}_{jH}, \quad H = A, B \tag{2}$$

where a curved over-bar denotes a known term.

The velocity consists of a translational velocity vector field $\mathbf{v}_G := \dot{\mathbf{u}}_G(s,t)$, a spin vector field $\boldsymbol{\omega} = \text{axial}[\dot{\mathbf{R}}(s,t)\mathbf{R}^T(s,t)]$, and a set of scalar velocity fields $\dot{a}_j(s,t)$. By time-differentiating the previous equations, we get the strain rates $\dot{\mathbf{e}}$, $\dot{\mathbf{k}}$, $\dot{\alpha}_j$, $\dot{\beta}_j$ as related to the stretching velocity gradients by [41]

$$\begin{aligned} &\mathbf{R}\dot{\mathbf{e}}_G = \mathbf{v}_G' - \boldsymbol{\omega} \times \mathbf{x}_G' \\ &\mathbf{R}\dot{\mathbf{k}} = \boldsymbol{\omega}' \\ &\dot{\alpha}_j = \dot{a}_j, \quad \dot{\beta}_j = \dot{a}_j' \end{aligned} \tag{3}$$

We consider the beam loaded by *generalized external forces*, defined as dynamic quantities spending virtual power on the independent velocity fields [42], via

$$\begin{aligned} \mathcal{P}_{ext} := &\int_{\mathcal{S}} \left(\mathbf{p} \cdot \mathbf{v}_G + \mathbf{c} \cdot \boldsymbol{\omega} + \sum_{j=1}^{n} q_j \dot{a}_j \right) ds \\ &+ \sum_{H=A}^{B} (\mathbf{P}_H \cdot \mathbf{v}_H + \mathbf{C}_H \cdot \boldsymbol{\omega}_H + Q_{jH} \dot{a}_{jH}) \end{aligned} \tag{4}$$

Here, $\mathbf{p}$, $\mathbf{P}_H$ and $\mathbf{c}$, $\mathbf{C}_H$ are forces and couples acting in the field and at the boundaries $H = A, B$ of the beam, respectively. Moreover, q_j, Q_j are terms peculiar to the locally deformable beam, referred to as *distortional forces*. Concerning the internal virtual power, we introduce *generalized internal forces*, or stresses, defined as follows:

$$\mathcal{P}_{int} := \int_{\mathcal{S}} \left(\mathbf{t} \cdot \mathbf{R}\dot{\mathbf{e}}_G + \mathbf{m} \cdot \mathbf{R}\dot{\mathbf{k}} + \sum_{j=1}^{n} (D_j \dot{\alpha}_j + B_j \dot{\beta}_j) \right) ds \tag{5}$$

where $\mathbf{t}$ and $\mathbf{m}$ are force-stress and couple-stress referred to the current configuration, respectively, and $\mathbf{R}\dot{\mathbf{e}}_G$, $\mathbf{R}\dot{\mathbf{k}}$ the pushed-forward strain-rates. Moreover, D_j, B_j are internal contact actions that will be called *distortional and bi-distortional stresses*, which are dual to the distortional strain-rates and their spatial gradients, respectively.

To obtain the balance equations, we equate the external and internal powers, and use the Virtual Power Principle (VP), by which the equality is satisfied for any admissible virtual motion. By using Eqs. (3), the Principle reads

$$\int_{\mathcal{S}} \left(\mathbf{p} \cdot \mathbf{v}_G + \mathbf{c} \cdot \boldsymbol{\omega} + \sum_{j=1}^{n} q_j \dot{a}_j \right) ds + \sum_{H=A}^{B} (\mathbf{P}_H \cdot \mathbf{v}_H + \mathbf{C}_H \cdot \boldsymbol{\omega}_H + Q_{jH} \dot{a}_{jH})$$