International Journal of Non-Linear Mechanics 66 (2014) 43–53

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Non-linear theories of beams and plates accounting for moderate rotations and material length scales

J.N. Reddy*, A.R. Srinivasa

Department of Mechanical Engineering, Texas A&M University, College Station, TX 77843-3123, USA

ARTICLE INFO

Article history:
Received 18 February 2014
Received in revised form
5 June 2014
Accepted 7 June 2014
Available online 17 July 2014

Keywords:
Beams and plates
Discrete peridynamics
Gradient elasticity
Material length scales
Modified couple stress theory

ABSTRACT

The primary objective of this paper is to formulate the governing equations of shear deformable beams and plates that account for moderate rotations and microstructural material length scales. This is done using two different approaches: (1) a *modified* von Kármán non-linear theory with modified couple stress model and (2) a gradient elasticity theory of fully constrained finitely deforming hyperelastic cosserat continuum where the directors are constrained to rotate with the body rotation. Such theories would be useful in determining the response of elastic continua, for example, consisting of embedded stiff short fibers or inclusions and that accounts for certain longer range interactions. Unlike a conventional approach based on postulating additional balance laws or ad hoc addition of terms to the strain energy functional, the approaches presented here extend existing ideas to thermodynamically consistent models. Two major ideas introduced are: (1) inclusion of the same order terms in the strain–displacement relations as those in the conventional von Kármán non-linear strains and (2) the use of the polar decomposition theorem as a constraint and a representation for finite rotations in terms of displacement gradients for large deformation beam and plate theories. Classical couple stress theory is recovered for small strains from the ideas expressed in (1) and (2). As a part of this development, an overview of Eringen's non-local, Mindlin's modified couple stress theory, and the gradient elasticity theory of Srinivasa–Reddy is presented.

1. An overview of theories with material length scales

1.1. Background

There has been increased interest in recent years in developing structural theories that have the ability to capture material length scale effects. This is primarily due to the need to model the structural response of a variety of new materials which are being developed that require the consideration of very small length scales over which the neighboring secondary constituents interact, especially when the spatial resolution (or length scale) is comparable to the size of the secondary constituents. Examples of such materials are provided by nematic elastomers and carbon nanotube composites [1] and environment resistent coatings made of CNT reinforced materials [2,3]. In addition, the flexoelectric effect [4], which is a size dependent strain gradient effect on the polarization of ferroelectrics, induces piezo-electric response in non-piezoelectric materials at very small scales.

Most structural systems involve the use of rods, beams, plates, and shells. They are also commonly used in micro- and nano-scale devices, that is, MEMS and NEMS. Due to the small physical dimensions of these devices, microstructure-dependent size effects are often exhibited by structural elements used in various micro- and nano-scale devices [5,6]. All beam and plate theories based on the classical elasticity theory do not account for the microstructure-dependent size effects. Therefore, the conventional beam and plate theories are not capable of predicting the size effects, that is, their response may be influenced by the microstructural parameters. Thus, it is useful to develop modified theories of beams and plates that account for size effects and geometric non-linearity. The present study is focused on formulating beam and plate theories with aforementioned effects. The following sections provide a background for the present study.

1.2. Eringen's non-local elasticity model

Classical continuum theories are based on hyperelastic constitutive relations which assume that the stress at a point is a function of strains at that point. On the other hand, the non-local continuum mechanics assumes that the stress at a point is a function of strains at all points, at least in some neighborhood of the point, in the continuum. These theories contain information about the forces between atoms, and the internal length scale is introduced into the constitutive equations as a material parameter.

* Corresponding author. Tel.: +1 979 862 2417; fax: +1 979 862 3989.
E-mail address: jnreddy@tamu.edu (J.N. Reddy).

http://dx.doi.org/10.1016/j.ijnonlinmec.2014.06.003

Such non-local elasticity was initiated in the works of Eringen [7–9] and Eringen and Edelen [10].

According to Eringen [7,8], the state of stress $\boldsymbol{\sigma}$ at a point $\mathbf{x}$ in an elastic continuum not only depends on the strain field $\boldsymbol{\varepsilon}$ at the point (hyperelastic case) but also on strains at all other points of the body. Eringen attributed this to the atomic theory of lattice dynamics and experimental observations on phonon dispersion. Thus, the non-local stress tensor $\boldsymbol{\sigma}$ at point $\mathbf{x}$ is expressed as

$$\boldsymbol{\sigma} = \int_{\Omega} K(|\mathbf{x}' - \mathbf{x}|, \tau)\mathbf{t}(\mathbf{x}')\, d\mathbf{x}' \quad (1)$$

where $\mathbf{t}(\mathbf{x})$ is the classical, macroscopic stress tensor at point $\mathbf{x}$ and the kernel function $K(|\mathbf{x}' - \mathbf{x}|, \tau)$ represents the non-local modulus, $|\mathbf{x}' - \mathbf{x}|$ being the distance (in the Euclidean norm) and τ is a material parameter that depends on internal and external characteristic lengths (such as the lattice spacing and wavelength, respectively). The macroscopic stress $\mathbf{t}$ at a point $\mathbf{x}$ in a Hookean solid is related to the strain $\boldsymbol{\varepsilon}$ at the point by the generalized Hooke's law

$$\mathbf{t}(\mathbf{x}) = \mathbf{C}(\mathbf{x}) : \boldsymbol{\varepsilon}(\mathbf{x}) \quad (2)$$

where $\mathbf{C}$ is the fourth-order elasticity tensor and : denotes the ‘double-dot product’ (see Reddy [11]).

The constitutive equations (1) and (2) together define the non-local constitutive behavior of a Hookean solid. Eq. (1) represents the weighted average of the contributions of the strain field of all points in the body to the stress field at point $\mathbf{x}$. In view of the difficulty in using the integral constitutive relation, Eringen [8] proposed an equivalent differential model as

$$(1 - \tau^2 \ell^2 \nabla^2)\boldsymbol{\sigma} = \mathbf{t}, \quad \tau = \frac{e_0 a}{\ell} \quad (3)$$

where e_0 is a material constant, and a and ℓ are the internal and external characteristic lengths, respectively.

The non-local theory of elasticity of Eringen has been used extensively in the last decade to study lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics, surface tension fluids, etc. The use of non-local elasticity to study size-effects in micro and nanoscale structures was first carried out by Peddieson et al. [12]. They used the non-local elasticity to study the bending of micro and nanoscale beams and concluded that size-effects could be significant for nano structures. Zhang et al. used non-local elasticity to show the small-scale effects on buckling of MWCNTs under axial compression [13] and radial pressure [14]. Wang [15] and Wang and Varadan [16] have studied wave propagation in carbon nanotubes (CNTs) with non-local Euler–Bernoulli and Timoshenko beam models. The small-scale effect on CNTs wave propagation dispersion relation is explicitly determined for different CNTs wave numbers and diameters by theoretical analyses and numerical simulations. The scale coefficient in non-local continuum mechanics is roughly estimated for CNTs from the obtained asymptotic frequency. The findings proved to be effective in predicting small-scale effect on CNTs wave propagation with a qualitative validation study based on the published experimental work. Wang et al. [17] formulated a non-local Timoshenko beam theory, neglecting the non-local effect in writing the shear stress–strain relation.

Reddy [18,19] and Reddy and Pang [20] have formulated non-local versions of various beam and plate theories, and presented numerical solutions of bending, vibration, and buckling of beams. The results show that the non-local parameter $\mu = \tau^2 \ell^2 = e_0^2 a^2$ has the effect of softening the beam, and thus predict larger deflections and lower buckling loads and vibration frequencies (see Figs. 1 and 2; taken from [18]).

There are numerous other papers that study the response of nanosystems using theories that are based on Eringen's differential model (see [21–24], and references therein). Reddy [18,19] pointed out that Eringen's differential model does not conform to the normal structural mechanics formulations in that the resulting equations are not derivable from a strain energy potential for the cases in which the von Kármán non-linearity along with kinetic energy are accounted for (see Reddy and El-Borgi [25] for details). Also, the force boundary conditions associated with the non-local beams, when expressed in terms of the displacement variables, contain the non-local parameter. Thus, there is a need to reexamine Eringen's differential model closely for its suitability in accounting for non-local elasticity.

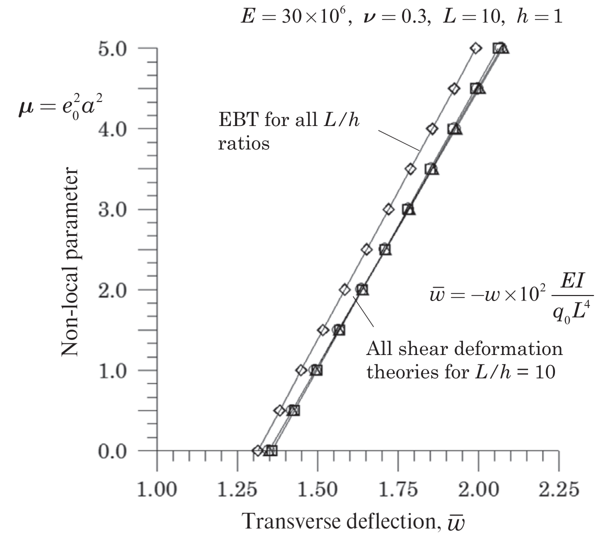

Fig. 1. Non-dimensional center deflection $\bar{w}$ vs. the non-local parameter μ for a simply supported beam under uniformly distributed load of intensity q_0 (L denotes the length and h is the height of the beam); taken from [18].

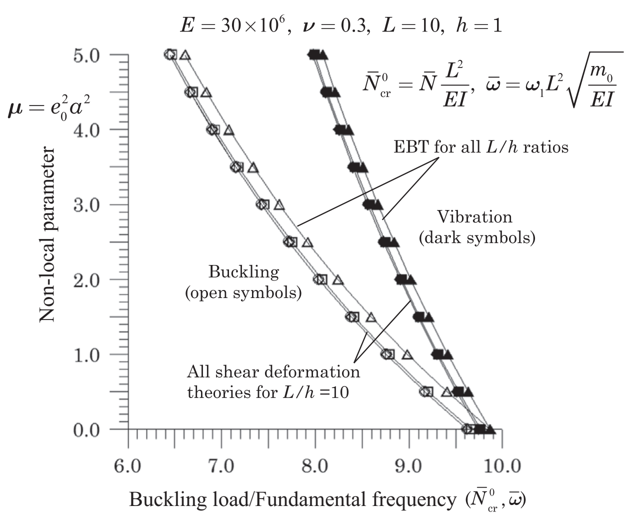

Fig. 2. Non-dimensional critical buckling load $\bar{N}_{cr}$ and non-dimensional natural frequency $\bar{\varpi}$ vs. the non-local parameter μ for a simply supported beam (L denotes the length and h is the height of the beam); taken from [18].

1.3. Modified couple stress theories

Theories of micro-structured media have been developed dating back to the 1960s. There is a large body of literature on small deformation couple stress theories with constrained micro-rotation, beginning with the early works of Mindlin, Toupin, Green, Naghdi, and Rivlin [26–32]. The classical couple stress elasticity theory of Koiter [27] contains four material length scale