International Journal of Non-Linear Mechanics 66 (2014) 105–110

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Geometric theory of smooth and singular defects

Marcelo Epstein [a,*], Reuven Segev [b]

[a] University of Calgary, Canada
[b] Ben-Gurion University of the Negev, Israel

ARTICLE INFO

Article history:
Received 17 November 2013
Received in revised form
15 February 2014
Accepted 17 February 2014
Available online 5 March 2014

Keywords:
Dislocations
De Rham currents
Differential geometry
Frobenius theorem
Frank's rules
Integrability

ABSTRACT

A unified theory of material defects, incorporating both the smooth and the singular descriptions, is presented based upon the theory of currents of Georges de Rham. The fundamental geometric entity of discourse is assumed to be represented by a single differential form or current, whose boundary is identified with the defect itself. The possibility of defining a less restrictive dislocation structure is explored in terms of a plausible weak formulation of the theorem of Frobenius. Several examples are presented and discussed.

1. Introduction

While the precise definition of the concept of material defect must be left to every particular context, a common feature of all theories dealing with defects (dislocations, inhomogeneity, and so on) appears to be that the presence of defects translates itself mathematically into the lack of integrability of some geometric entity. In a general differential geometric framework, questions of integrability pertain to differential forms and their exactness or lack thereof. It seems appropriate, therefore, to undertake a unified treatment of defects by associating to any possible structure under consideration one or more differential forms. On the other hand, since differential forms are, by definition, smooth entities, it would appear that the rich variety of isolated defects, whose practical and historical importance cannot be denied, might be left out and that a unified treatment encompassing both the continuous and discrete cases would remain out of reach of a single formal apparatus. The situation is similar in many other engineering applications, where concentrated entities (forces, masses, charges) can be seen as limiting cases of their smooth counterparts. The unified mathematical treatment of these cases was historically achieved by the theory of distributions, where the singular entities are represented not by functions but rather by linear functionals on a suitable space of test functions. The most common example is provided by the Dirac delta which assigns to each compactly supported smooth function in $\mathbb{R}$ its value at the origin. Since a scalar field can be considered as a particular case of a differential form, it is not surprising that Schwartz's theory of distributions can be extended to forms of all orders. This extension, achieved by de Rham [1], is completely general and independent of any metric considerations, a feature that should be considered essential in a truly general geometric setting. de Rham introduced the terminology of *currents* to designate his generalized differential forms. It is this tool that will serve our purposes in the present formulation of the unified theory of defects.[1]

2. Currents

2.1. Definition

A p-current on an n-dimensional manifold $\mathcal{M}$ is a continuous[2] linear functional $T[\phi]$ on the vector space of all C^∞ p-forms ϕ with compact support in $\mathcal{M}$. To understand in what sense this definition is consistent with that of smooth forms, it suffices to exhibit the latter as a particular case of the former. Let, therefore, ω represent a smooth p-form on $\mathcal{M}$. We can uniquely associate

* Corresponding author.
E-mail addresses: mepstein@ucalgary.ca (M. Epstein), rsegev@bgu.mail.il (R. Segev).

[1] This work extends our previous article [2].

[2] By continuity we mean that the sequence of evaluations $T[\phi_i]$ on a sequence of C^∞ p-forms supported in a coordinate neighborhood within a common compact subset of $\mathcal{M}$ tends to zero whenever the coefficients of a coordinate representation of the forms ϕ_i and all their derivatives tend to zero uniformly as $i \to \infty$.

http://dx.doi.org/10.1016/j.ijnonlinmec.2014.02.006
0020-7462

to it the $(n-p)$-current T_ω defined as the linear operator

$$T_\omega[\phi] = \int_{\mathcal{M}} \omega \wedge \phi, \tag{1}$$

for all $(n-p)$-forms ϕ with compact support in $\mathcal{M}$. Strictly speaking, the $(n-p)$-current T_ω cannot be "equal" to the p-form ω, but they are indistinguishable from each other in terms of their integral effect on all "test forms" ϕ. Thus, a form bears to its associated current the same relation that a function bears to its associated distribution. An important non-trivial example of a current that is not associated to any differential form is the following. Let s be a p-simplex in $\mathcal{M}$. We associate to it the p-current defined by

$$T_s[\phi] = \int_s \phi, \tag{2}$$

for all compactly supported p-forms ϕ. The definition above can be extended by linearity to arbitrary chains. These examples show how an integrand and a domain of integration are unified under the single formal umbrella of currents.

2.2. Operations

1. Currents of the same dimension can be added together and multiplied by real numbers in an obvious way.
2. The inner product of a p-current T with a q-form α is the $(p-q)-$current $T \llcorner \alpha$ defined as

$$(T \llcorner \alpha)[\phi] = T[\alpha \wedge \phi]. \tag{3}$$

Similarly,

$$\alpha \lrcorner T = (-1)^{(n-p)q}\; T \llcorner \alpha. \tag{4}$$

3. The product of a p-current T with a vector field $\mathbf{X}$ is the $(p+1)-$current :

$$(T \wedge \mathbf{X})[\phi] = T[\mathbf{X} \lrcorner \phi], \tag{5}$$

where $\mathbf{X} \lrcorner \phi$ is the contraction of the form ϕ with the vector field X.
4. The boundary of a p-current T is the $(p-1)-$current

$$\partial T[\phi] = T[d\phi]. \tag{6}$$

Using Stokes' theorem for chains, it is easy to show that, for any chain c,

$$\partial T_c = T_{\partial c}. \tag{7}$$

2.3. Possibilities

The notion of current opens the doors for a generalization of classically smooth differential geometric objects, such as connection, torsion and curvature. While this idea is beyond the scope of this paper, it is not difficult to intuit the possibilities. Consider, for example, a (non-zero) decomposable differential p-form ω on an n dimensional differentiable manifold $\mathcal{M}$. Thus, there exist p linearly independent 1-forms ω_i $(i = 1, \ldots, p)$ such that

$$\omega = \omega_1 \wedge \cdots \wedge \omega_p. \tag{8}$$

Such a form uniquely determines at each point of $x \in \mathcal{M}$ an $(n-p)-$dimensional subspace H_x of the tangent space $T_x\mathcal{M}$. A vector $\mathbf{v} \in T_x\mathcal{M}$ belongs to H_x if $\omega_i(\mathbf{v}) = 0$ for each $i = 1, \ldots, p$. The collection $\mathcal{H}$ of all the subspaces H_x is called a (geometric) $(n-p)$-dimensional *distribution* on $\mathcal{M}$. Conversely, given a distribution, the corresponding decomposable form ω is determined up to multiplication by a scalar field $\alpha : \mathcal{M} \to \mathbb{R}$. A submanifold $\mathcal{S}$ of $\mathcal{M}$ is called an *integral manifold* of the distribution $\mathcal{H}$ if for every $s \in \mathcal{S}$ we have $T_s\mathcal{S} = H_s$. A distribution is *completely integrable* if at every point x it admits an integral manifold of maximal dimension (i.e., $n-p$). According to one of the versions of the theorem of Frobenius, a distribution $\mathcal{H}$ defined by a decomposable form ω is completely integrable if, and only if, there exists a 1-form β such that

$$d\omega = \beta \wedge \omega. \tag{9}$$

This is tantamount to saying that, for some choice of the scalar degree of freedom α, the form $\alpha\omega$ is closed, namely, there exists an integrating factor α such that $d(\alpha\omega) = 0$.

So far, we have been dealing with the smooth case only. Assume now that we have a means of rigorously defining and characterizing the decomposability of a current. In that case, we could declare that a decomposable p-current T determines a p-dimensional *singular geometric distribution* on $\mathcal{M}$ and we could define the complete integrability of the singular distribution by the condition

$$\partial T = \beta \lrcorner T, \tag{10}$$

for some 1-form β.[3] A stronger condition would be to require that β be closed.

Notice that since connections in general can be regarded as (horizontal) distributions on fibre bundles, and since the curvature of a connection is related to its complete integrability, we can expect that singular connections can be introduced by means of decomposable currents and their non-vanishing curvature can be detected by the violation of a condition such as (10). By this means, a situation is envisioned in which the standard curvature vanishes almost everywhere and is concentrated, as it were, at a single point. We remark that Eq. (10) is by no means the result of a theorem, but only a possible definition of complete integrability of a singular distribution. Clearly, the fact that β is a smooth form may severely limit the singular distributions that can be considered completely integrable.

3. Bravais hyperplanes

3.1. The smooth case

The traditional heuristic argument to introduce continuous distributions of dislocations in crystalline materials calls for the specification of a *frame field* (or *repère mobile*) in the body manifold, and the consequent distant parallelism.[4] An alternative, dual, picture is obtained by means of a *co-frame field*, which can be regarded as an $\mathbb{R}^n$-valued 1-form on $\mathcal{M}$. This point of view suggests perhaps that n linearly independent 1-forms might constitute a convenient point of departure for our desired generalization. Each of these 1-forms would represent a family of Bravais planes. It comes as a surprise, however, that defects are meaningful and detectable with just a single family of such planes or, more specifically and less surprisingly, that integrability conditions can be associated with a single 1-form on a manifold.

Geometrically, a 1-form (always decomposable) induces an $(n-1)$-dimensional distribution, that is, a field of hyperplanes. It is physically important to point out that, relinquishing the

[3] As a curiosity, it is interesting to remark that Eq. (10) can be informally regarded as the eigenvalue problem of the boundary operator ∂. Its "eigenvectors" are the completely integrable currents.

[4] This feature is present also, albeit with the degree of freedom afforded by material symmetries, in the constitutively based approach propounded by Kondo [3] and Noll [4], whereby points are compared, in a groupoid-like fashion, via material isomorphisms between their tangent spaces.