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On the corner behavior of a non-linear elastic wedge under mixed boundary conditions

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ABSTRACT

This study concerns the local behavior of the solutions of the governing equations of non-linear elastostatics in the vicinity of the corner of a wedge-shaped region of angle $\alpha \in (0, 2\pi]$. It contains an asymptotic investigation in the plane strain regime – using a subclass of non-linear harmonic elastic solids that have a proper asymptotic constitutive structure in the vicinity of the corner – of the deformation field near the corner point that separates a free from an adjoining fixed segment of the boundary. It is well-known that, as the corner point is approached, the singular field behavior predicted by the classical linear theory of elasticity yields oscillatory deformations that are not injective. This anomalous behavior is related to a spurious self-intersection anomaly. Using a proper constitutive structure within a subclass of non-linear harmonic materials in the vicinity of the corner, we obtain an asymptotic expansion of the deformation field for which the foregoing anomalous behavior is avoided. The conditions on this field which guarantee injectivity lead to an unexpected behavior of the deformed free surface that is physically possible. In the particular case of $\alpha = \pi$, the second-order expansion of the deformation field agrees with its counterpart found elsewhere in the literature. For another particular case, concerning $\alpha = \pi/2$, the second-order expansion is an improvement over its counterpart found in the literature.

1. Introduction

This study concerns the local behavior of the solutions of the governing equations of non-linear elastostatics in the vicinity of the corner of a wedge-shaped region of angle $\alpha \in (0, 2\pi]$. It contains an asymptotic investigation in the plane strain regime of the deformation field near the corner point that separates a free from an adjoining fixed segment of the boundary. It is well-known that, as the corner point is approached, the singular field behavior predicted by the classical linear theory yields oscillatory deformations that are not injective.[1] This oscillatory behavior is related to a spurious self-intersection anomaly.

Our goal in this paper is to use a representative non-linear theory of elasticity to gain understanding of the behavior of the deformation field in the vicinity of points on the boundary of an elastic body where either the boundary, the boundary conditions, or both are not smooth. In particular, stress–displacement boundary conditions are imposed on either side of the vertex of the wedge-shaped region.

Here, we concentrate on the particular class of *harmonic materials* introduced by John [2], which are hyperelastic, compressible, and isotropic solids. Knowles and Sternberg [3] are the first to use this class of materials in the investigation of the singular behavior of solutions in the vicinity of points where the boundary conditions are not smooth. For this, they consider the *plane bonded punch problem*, which concerns the equilibrium of a half-plane in the absence of body force that arises from the requirement that a finite segment of the boundary undergoes a constant rigid normal translation by means of an axially loaded, bonded flat-ended rigid punch. The remainder of the boundary, as well as the points at infinity, are free of traction. Assuming that the sum of the principal stretches, ω_1, tends to infinity as a corner of the punch is approached, these authors use a subclass of harmonic materials with a proper asymptotic constitutive structure that yields asymptotic expressions for the deformation gradients having positive determinant everywhere in a neighborhood of the corner. This asymptotic structure implies that a rectangular piece of the material subject to homogeneous uni-axial tension maintains a non-zero width for any finite but arbitrarily large uni-axial stretch. The asymptotic structure depends on two parameters, which are denoted by γ_1 and γ_2 in this work, that satisfy the inequalities

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[1] See, for instance, Aguiar and Fosdick [1] for details about this anomalous behavior and related references.

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$0 < \gamma_1 < 1$ and $\gamma_2 > 0$ and are, otherwise, not specified. Of course, these parameters are not arbitrary, being dependent on the Lamé constants λ and μ of the linear theory. To obtain the asymptotic expressions, the authors assume that the deformation field has a certain radial dependence as the corner is approached and solve ordinary differential equations to determine the azimuthal dependence of this field.

Later, Aguiar and Fosdick [1] combine the above asymptotic constitutive structure for large ω_1 with the *semi-linear material* introduced by John [2] for ω_1 not large and investigate the behavior of solutions of the plane bonded punch problem in the neighborhood of the corners of the punch. The resulting material model is called *modified semi-linear material*. For this model, we obtain expressions for the parameters γ_1 and γ_2 of the asymptotic structure in terms of the Lamé constants λ and μ. We have used the modified semi-linear material together with the finite element method to simulate numerically the behavior of the non-linear elastic solid in the vicinity of the corners of the punch and provided independent computations that confirm the free surface behavior predicted by the asymptotic expressions obtained by Knowles and Sternberg [3]. In particular, both asymptotic and computational results predict a non-monotonic behavior of the deformed free surface for any applied axial load.

To see this unusual behavior more clearly, we consider that, for zero axial load, a modified semi-linear solid occupies the right-hand side of a vertical plane containing the surface of the punch. We then show in both Fig. 6a and b of Aguiar and Fosdick [1] the deformed free surfaces of the solid at the scale distance $O(10^{-10}L)$, where $2L$ is the width of the punch, from a corner of the bonded punch for small (Fig. 6a) and larger (Fig. 6b) axial loads according to both the asymptotic and the computational predictions. On each frame of Fig. 6, the thick white line represents the punch surface and the black line represents the scale distance in terms of the width of the punch. On the right side of each frame, the gray scale represents values of the Jacobian determinant $J \overset{\text{def}}{=} \det \mathbf{F}$, where $\mathbf{F}$ is the deformation gradient. Based on this gray scale, we compare the values of J on the free surface obtained from the asymptotic representation of the modified semi-linear constitutive assumption with the corresponding values obtained from our computational scheme. Values of J greater than $+2$ are represented by the white color and values of J smaller than -2 are represented by the black color. While an everywhere positive determinant is acceptable, the existence of negative values is related to self-intersection and keynotes a deficiency in the constitutive model. A plot of the indenting axial punch load P versus the horizontal displacement U of the punch is shown at the upper right corner of each frame. The deformed free surfaces predicted by both the asymptotic and the computational results lie on top of one another in both frames, Fig. 6a and b, and show a curious non-monotonic behavior of the free surface, which concerns the deformed free surface being depressed farther to the right side of the flat punch surface near the corner. Away from the corner, the computations predict that, albeit not shown in the figure, the free surface deforms to the left of the indenting punch, as expected. In this work, a similar behavior is predicted by our asymptotic analysis in a neighborhood of the vertex of the wedge for wedge angles different than π. We comment more on this unexpected behavior below.

Recently, Kim et al. [4] have extended the work of Knowles and Sternberg [3] to the case of a quarter-plane region occupied by a harmonic material with the asymptotic constitutive structure mentioned above and subjected to zero traction and displacement conditions on either side of the corner. Recall from above that the constitutive parameters of the asymptotic structure satisfy the inequalities $0 < \gamma_1 < 1$ and $\gamma_2 > 0$ and are, otherwise, not specified. The authors show conditions that guarantee no self-penetration in the vicinity of the corner.

Here, we use the modified semi-linear material to investigate the behavior of the deformation field in the vicinity of the vertex of the wedge-shaped region subjected to zero traction and zero displacement conditions on either side of the vertex. Away from the corner, on a circular arc of angle α, we assume that the deformation field is smooth and known from the solution of the equilibrium problem for the whole elastic body. For instance, in the plane bonded punch problem described above, the angle of the wedge-shaped region is π and, without loss of generality, we superimpose a rigid body displacement on the whole elastic body so that the finite segment on the boundary is fixed in the mathematical formulation of the problem, yielding the homogeneous displacement on one side of the vertex of the wedge. Following Ogden and Isherwood [5], we then use a complex variable formulation of the governing equations for the plane-strain deformation of the harmonic solid occupying this region and present general expressions for the stress and deformation fields in terms of two analytic functions. In classical linear elasticity, analogous expressions are given by the classical Kolossoff's formulae for the stress and displacement fields. The similar structure of these expressions in the linear and non-linear theories allows us to infer the asymptotic expansion of the solution of the non-linear problem based on the asymptotic expansion of its linear counterpart. To see this, we present below a few results from the classical linear theory.

Bogy [6,7] and Reitich [8] consider the singular behavior of the displacement field in the vicinity of the apex of two materially dissimilar wedge-shaped regions of arbitrary angles that are bonded together along a common edge and subjected to surface tractions on the boundary within the plane theory of linear elastostatics. In particular, Reitich [8] applies the general theory of elliptic boundary value problems in domains with isolated singular points in the analysis of this wedge problem and presents the general expression of the displacement field in the vicinity of the apex. The singular terms of this expression that gives rise to unbounded stresses are of the form

$$r^{-ib}(\log\, r)^k \psi(\theta) \tag{1}$$

in the vicinity of the corner, where (r, θ) are polar coordinates with origin at the corner, ψ is an infinitely differentiable function, and b is a complex root with multiplicity $k = 0, 1, \ldots,$ of a certain transcendental equation $\Upsilon(b) = 0$. The number of singular terms is determined by the number of roots of the transcendental equation for which $0 < \operatorname{Im} b < 1$. If the Lamé constant λ is positive, the transcendental equation has no more than three zeroes (counted with multiplicities). However, if $\alpha = \pi/2$, then there is no more than one zero, and for all the cases computed numerically by Reitich [8], the number of zeroes never exceeded two.

In this work, we take advantage of the similar overall structure of the solutions of both linear and non-linear problems and assume that the asymptotic expansions of the two analytic functions in the non-linear case are similar to their counterparts in the linear case, which are of the form (1). We show that the asymptotic behavior of the deformation field is determined from the solution of a system of algebraic equations that are obtained from the imposition of boundary conditions on either side of the corner. We then find that, depending on the value of α, none $(\alpha < \pi/2)$, one $(\pi/2 < \alpha < 3\pi/2)$, or, two $(3\pi/2 < \alpha \le 2\pi)$ singular terms can be expected in the asymptotic expansion of the deformation field. If $\alpha = \pi/2$, or, $\alpha = 3\pi/2$, then the asymptotic expansion contains logarithmic terms.

Albeit different from the approach used by Knowles and Sternberg [3] and Kim et al. [4], which requires the solution of ordinary differential equations, our approach, valid for all $\alpha \in (0, 2\pi]$, yields the same second-order asymptotic approximation of the deformation field obtained by the first author for the